(ii) *If a number is a multiple of 4, it is also a multiple of 8.*

(iii) *If a number is a multiple of 9, it is also a multiple of 2.*

(iv) *Multiples are positive integers.*

b) *Is it always, sometimes or never true that adding two consecutive multiples of 5 will give a multiple of 10?*

c) *Is it always, sometimes or never true that adding five consecutive multiples of 2 will give a multiple of 10?*

Students who have demonstrated a secure understanding of identifying multiples should be encouraged to go deeper by solving more complex problems, such as exploring all possibilities, creating their own examples and testing conjectures. Students should solve problems where there is more than one answer and there are elements of experimentation, investigation, checking, reasoning, proof, etc.

It is important that students are given opportunities to investigate multiplicative relationships. *Example 2* provides a structure for this as well as an opportunity to reason and use examples and counter-examples to demonstrate their mathematical understanding.

Example 3: *Two lighthouses flash at different intervals. One flashes every 5 seconds and the other every 8 seconds.*

At exactly midnight (00:00:00) they flash together. When will they next flash at the same time?

Example 3 provides students with an opportunity to apply their mathematical understanding to unfamiliar problems and to practise their understanding of a concept (i.e. intelligent practice rather than mechanical repetition) through focusing on relationships, rather than the procedure.

Use the prime factorisation of two or more positive integers to efficiently identify the highest common factor

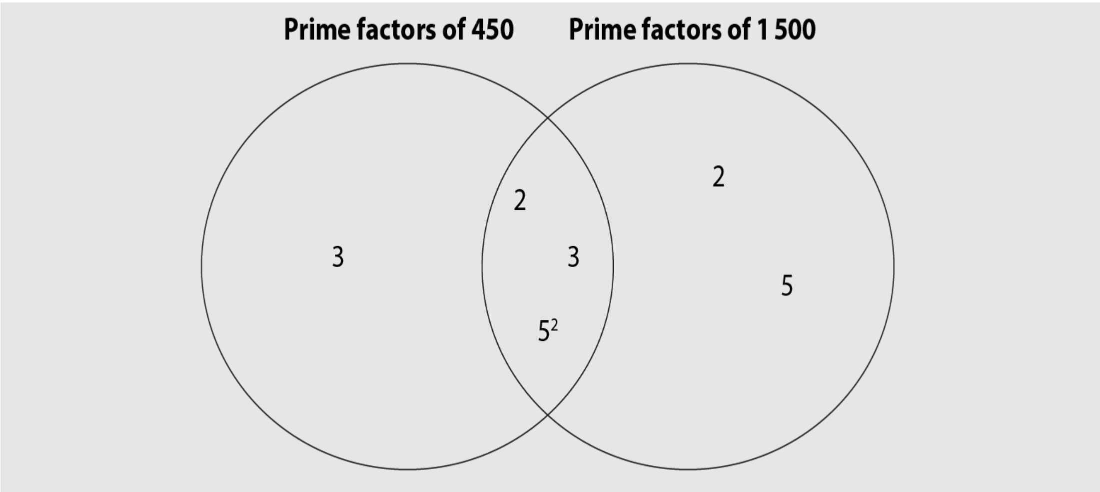
Common difficulties and misconceptions*:* fundamental to this concept is that students are at ease with the idea that 2 × 3 × 5 is just another way of expressing the number 30 and does not need to be calculated. In fact, by leaving it in this form, it is much easier to discern factors and (when there are two numbers expressed in this way) to discern their common factors. However, students often struggle with this idea. When asked whether 2 × 3 × 5 is a multiple of 10 or not, it is not uncommon for students to multiply the three factors together to obtain 30 before they are able to say that it is a multiple of 10.

An important awareness in this key idea is that when two numbers are written as the product of two or more factors, looking for overlaps between the two products helps to find a common factor. For example, by writing 30 = 5 × 6 and 105 = 5 × 21, it is easily seen that five is a common factor.

However, we cannot be sure that five is the *highest* common factor unless each number is written as the product of prime factors, e.g. 30 = 2 × 3 × 5 and 105 = 3 × 5 × 7. When written in this format, it is clear that 3 × 5 = 15 is the highest common factor.

Students may experience difficulties when the product of repeated prime factors is expressed using index notation (e.g. $450 = 2 \times 3^2 \times 5^2$ and $1\,500 = 2^2 \times 3 \times 5^3$), as it may be harder to detect the common factor of $2 \times 3 \times 5^2$ in this form. It will be important for students to have experience of discerning highest common factors from both the index and non-index form to help avoid this difficulty.

Students will have already experienced Venn diagrams in Key Stage 2. However, using them to record prime factors, and for revealing that the highest common factor is the product of the prime factors in the intersection, is likely to be an unfamiliar idea. The example above could be displayed in the Venn diagram below, showing that the highest common factor of 450 and 1 500 is $2 \times 3 \times 5^2$, or 150.



$450 = 2 \times 3^2 \times 5^2$ $1\,500 = 2^2 \times 3 \times 5^3$

Exploring multiple methods (such as listing factors, using the prime factorisation, using a Venn diagram, etc.) and establishing which is most efficient for numbers of varying sizes is important in this key idea. Discussing and comparing different approaches and solutions will support students in identifying and choosing appropriate and efficient methods.

Encourage students to apply this idea to algebraic expressions, for example:

What is the highest common factor of p^2qr^3 and pq^2r^2?

This will support them in generalising this idea and developing a deep and secure understanding of the underpinning mathematical structure.

Example 1: *Find the highest common factor of these pairs of numbers.*

a) *10 = 2 × 5*
   6 = 2 × 3

b) *12 = 2 × 2 × 3*
   20 = 2 × 2 × 5

c) $30 = 2 \times 3 \times 5$
$70 = 2 \times 5 \times 7$

d) $60 = 2 \times 2 \times 3 \times 5$
$90 = 2 \times 3 \times 3 \times 5$

e) $42 = 2 \times 3 \times 7$
$210 = 2 \times 3 \times 5 \times 7$

f) $29 = 29$

$16 = 2 \times 2 \times 2 \times 2$

Choosing small numbers allows students to find the highest common factor using multiple methods, both by listing factors of each number and by considering prime factors. In *Example 1*, the numbers have already been written as a product of their prime factors so that students can focus on finding the highest common factor and not on prime factorisation, although they should realise that this is a previous step in this method.

Part of the purpose of an exercise like this is for students to explore the most efficient method and understand why, when numbers are written as product of prime factors, the combination of their common prime factors will generate the highest common factor.

- Part *a* has been written so that the numbers only share one common prime factor (2).
- Part *b* is designed to ensure that students can identify and understand that the numbers share two common prime factors. Comparing the repeated common prime factor of two, to the common factors from the lists (2 and 4), is also worthwhile so that students understand that it is the combination of the prime factors that is important.
- In part *c*, the two common prime factors are different and lead to a highest common factor of 10. Again, comparison with common factors of 2, 5 and 10 from their list of factors is important.
- The numbers in part d have been chosen as they share three common prime factors.
- In part *e*, the highest common factor is one of the numbers in its entirety.
- Part *f* has been chosen to facilitate discussion about what to do if the highest common factor is 1, as this is not always apparent when numbers are written as the product of their prime factors.

Example 2:

a) Gosia thinks the highest common factor of 72 and 180 is 6. Her working is below:

$72 = 2^3 \times 3^2$

$180 = 2^2 \times 3^2 \times 5$

$HCF = 2 \times 3 = 6$

Do you agree with Gosia? Justify your answer.

b) $x = a^2 \times b \times c^2$
$y = a \times b^3 \times c^2$

where a, b and c are prime.

Harrison thinks the highest common factor of x and y is $a \times b \times c^2$.

Do you agree with Harrison? Justify your answer.

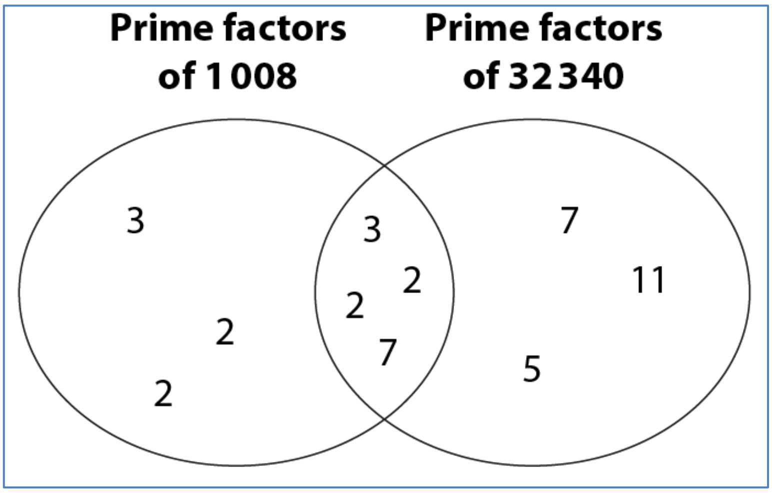
In *Example 2,* in part a the numbers are large enough to encourage students to move away from listing factors and instead compare prime factors to find the highest common factor. Asking students to find all common prime factors by deconstructing the simplified products may support them in finding the highest common factor.

In part b, the concept has been applied to algebraic prime factors. Having discussed part a, students then have the opportunity to demonstrate understanding in a more generalised form.

Challenge students to think deeply about the concept by asking them to create questions of their own that meet specific criteria. For example:

- Write down two numbers which have a highest common factor of…
- Write down two numbers which have a highest common factor of 1
- $72 = 2^3 \times 3^2$
  $180 = 2^2 \times 3^2 \times 5$
  - What is the highest common factor of 72 and 180?
  - Find an additional number so that the highest common factor of the three numbers (72, 180 and the new number) remains the same.
  - Write down the prime factorisation of a third number so that all three numbers have a highest common factor that is less than 12.
  - Write down a number with a highest common factor of 12 when paired with 72, but a highest common factor that is greater than 12 when paired with 180.

Example 3:



a) (i) Use the Venn diagram to write 1 008 and 32 340 as products of their prime factors.

(ii) Use the Venn diagram to find the highest common factor of 1 008 and 32 340.

b) (i) 1 575 = $3^2 \times 5^2 \times 7$

2 310 = $2 \times 3 \times 5 \times 7 \times 11$

Show the prime factors of 1 575 and 2 310 in a Venn diagram.

(ii) Use this Venn diagram to find the highest common factor of 1 575 and 2 310.

c) (i) Show the prime factors of 165 and 385 in a Venn diagram.

(ii) Use this Venn diagram to find the highest common factor of 165 and 385.

d) Use a Venn diagram to find the highest common factor of 150, 60 and 138.

It is important that students are familiar with a range of different representations and understand that, once products are expressed as prime factors, these can be shown clearly in a Venn diagram.

Students should explore how the intersection of sets in a Venn diagram clearly shows all common factors, which can then be used to find the highest common factor.

- In part *a*, students are given a completed Venn diagram and asked to use the information shown to write two numbers as products of prime factors. By doing this, students are demonstrating that they know what the Venn diagram is showing. Students then need to find the highest common factor using the product of the common prime factors shown in the intersection.
- In part *b*, students are given the numbers in prime factorisation form, but not given the Venn diagram – they must construct it from the information provided and then go on to find the highest common factor.
- Part *c* progresses to ask students to draw a Venn diagram from scratch and then use it to find the highest common factor.
- Part *d* asks students to find the highest common factor of three numbers using a Venn diagram.

This scaffolding should support students to ensure that all can progress together through each step as a class.