Transformations

Overview

Transformations describe different ways of mapping points on a plane to other points on the plane. A way to think about, describe and classify transformations is to consider what changes and what stays the same under different transformations. This also allows for discussion about congruence and similarity.

At Key Stage 2, students will have encountered all four transformations – translation, reflection, rotation and enlargement – and learnt to distinguish between them. However, they may not have concentrated on specific features, such as the centre of rotation or the centre of enlargement.

In all four transformations, students should recognise that every element of the object, i.e. every point, line or curve, or interior space, etc., undergoes the same transformation and that looking at each of these elements in turn will help them to accurately construct the image.

Dynamic geometry software offers an effective tool to support the teaching of transformations. It enables students to see what happens when certain transformations are applied to objects; and to make conjectures, justify and test where, for example, the image of an object under a reflection will be.

The order in which transformations have been introduced in this work– translation, rotation, reflection and, finally, enlargement – highlights how the degrees of freedom available, with regards to what can vary, are being increased. Translation maintains congruence and orientation. Rotation produces a change in orientation but maintains the 'sense' of the image – a feature which is able to change only under reflection. Translation, rotation and reflection produce congruent shapes in an increasing range of orientations and senses. Enlargement is the only transformation that does not maintain congruence (other than when the scale factor is ±1) but does maintain similarity in any orientation and sense.

In this set of key ideas, it will be useful for students to consider what's the same and what's different about an object and its image as they work on different transformations.

Prior learning

Before beginning transforming shapes at Key Stage 3, students should already have a secure understanding of the following learning outcomes from study at upper Key Stage 2:

- Identify, describe and represent the position of a shape following a reflection or translation, using the appropriate language, and know that the shape has not changed.
- Draw and translate simple shapes on the coordinate plane and reflect them in the axes.

- Solve problems involving similar shapes where the scale factor is known or can be found.

The NCETM has created the following Key Stage 2 ready-to-progress criteria to support teachers in making judgements about students' understanding and knowledge.

4G-1 Draw polygons, specified by coordinates in the first quadrant, and translate within the first quadrant.

4G-3 Identify line symmetry in 2D shapes presented in different orientations. Reflect shapes in a line of symmetry and complete a symmetric figure or pattern with respect to a specified line of symmetry.

Checking prior learning

The following activities from the NCETM primary assessment materials offer useful ideas for teachers to use to check whether prior learning is secure.

Reference	Activity
Year 6 page 36	*Are these statements always, sometimes or never true?* • *If a shape is reflected in an axis, it stays in the same quadrant.* • *If a shape is translated to the right and up, it stays in the same quadrant.* • *If a shape is translated to the left and down, it stays in the same quadrant.* *Explain your decisions.*

Language

centre of enlargement, centre of rotation, congruent (figures), enlargement, image, object, scale factor, similar

Progression through key ideas

Understand and use translations

When an object undergoes a translation, the size of its angles and the lengths of its lines are maintained so that the object and image are congruent. This property is shared with both rotation and reflection, but a translation, uniquely, always maintains the orientation of the object in the image. The use of notation to record a translation may follow from a need to describe it accurately and succinctly. Initially, students are likely to use informal language as they develop their understanding of the transformation: describing, for example, a move of 'three across' and 'two down'. While the formal use of vectors is part

of the national curriculum Key Stage 4 programme of study, translation offers students a natural opportunity to formalise their intuitive understanding about the distinction between movement and position.

Key ideas

- Understand the nature of a translation and appreciate what changes and what is invariant
- Understand the minimum information required to describe a translation (vertical and horizontal displacement)
- Translate objects from information given in a variety of forms

Understand and use rotations

As with translations, rotations maintain congruence but offer a further degree of change between the object and the image, since the orientation of the object is not necessarily maintained. In Key Stage 2, students will have worked with objects rotated through a half, a quarter and three-quarters of a turn. This is generalised to any angle at Key Stage 3, specifying the size and direction of turn. In addition, more attention is paid to the centre of rotation (the one point which does not move under the rotation) and the fact that the position of the image changes with different centres of rotation, even though the orientation may not.

In the construction of examples to support students' understanding, it is important to vary the position of the centre of rotation to include:

- on a vertex of the object
- lying within the object
- lying outside of the object.

Key ideas

- Understand the nature of rotations and appreciate what changes and what is invariant
- Understand the minimum information required to describe a rotation (centre of rotation, size and direction of rotation)*
- Rotate objects using information about centre, size and direction of rotation

Understand and use reflections

Transforming an object by reflecting it offers the full range of possible congruent shapes, and a context in which congruence may be explored further. Reflection in lines which are neither horizontal nor vertical presents increased challenge and requires students to have a sense of where the image will be. Using a range of tools, such as dynamic geometry software, alongside pencil and paper methods, gives students a greater depth of understanding.

Key ideas

- Understand the nature of reflections and appreciate what changes and what is invariant

- Understand the minimum information required to describe a reflection (line of reflection)*
- Reflect objects using a range of lines of reflection (including non-vertical and non-horizontal)

Understand and use enlargements

Students are likely to be familiar with enlargements through their work on similar shapes in Key Stage 2. At Key Stage 3, they are introduced to the idea of a centre of enlargement and that the position of this in relation to the object affects the image's position. In this set of key ideas, students consider the range of possible outcomes with an enlargement. They should come to appreciate that enlargement is the only transformation that does not guarantee a congruent shape.

At Key Stage 3, the focus is on enlargements with a scale factor ≥1, but the use of dynamic geometry software offers students an opportunity to reason mathematically about the images that will result if a scale factor outside of this range is used (as it is in Key Stage 4), and to then test and refine their conjectures.

Key ideas

- Understand the nature of enlargements and appreciate what changes and what is invariant
- Understand the minimum information required to describe an enlargement (centre of enlargement and scale factor)
- Enlarge objects using information about the centre of enlargement and scale factor

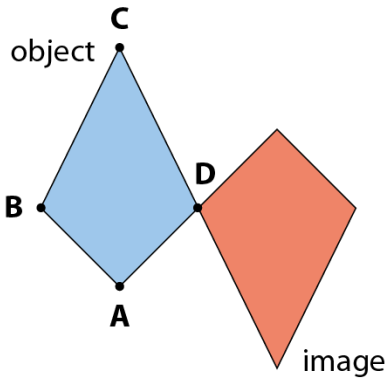
Exemplified significant key ideas

Understand the minimum information required to describe a rotation (centre of rotation, size and direction of rotation)

Common difficulties and misconceptions: a rotation is arguably one of the more challenging transformations for students to visualise. Plenty of opportunity to experiment and become familiar with the behaviour of rotations – for example, through the use of dynamic geometry software and hands-on activities using cut-out shapes or tracing paper – may support students in better understanding rotations.

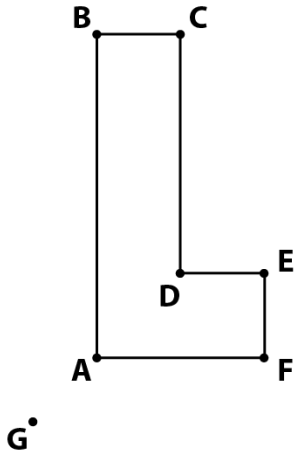
Describing and recording a rotation can prove challenging for students, since it draws on their understanding of angle as a measure of turn. Research suggests that this understanding of angle is not common in students in early Key Stage 3. Rather, an angle is often viewed only as a static measure of the relationship between two lines, i.e. a measure of 'pointedness' (Mitchelmore & White, 2000). Rotating several key points or elements of an object to obtain the image may help to establish a better understanding. Examples are given below.

Example 1: *The image is a rotation of the object. Describe two possible rotations that transform the object to this image.*



Rotations require three pieces of information to be fully described: a centre of rotation, a size of rotation and a direction of rotation. In *Example 1*, students are invited to imagine rotating the object around the point D. Whole-class discussion may reveal that some students rotated the object clockwise, while others rotated it anticlockwise. Students could also consider the reverse transformation. That is, how would the description of the rotation differ if the object and the image were exchanged?

Example 2:



Using tracing paper, find the position of the image if this object is rotated 90° clockwise about:

a) A
b) D
c) F
d) G

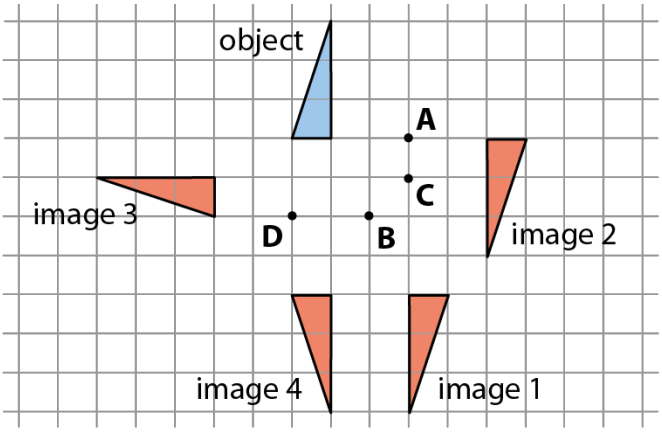
What's the same and what's different about these four images?

In *Example 2*, the centre of rotation has been varied to being separate from the object, but all other features of the rotation are the same. This should draw students' attention to the following key points:

- The orientation of the image is the same wherever the centre of rotation is.
- Once one point and one line have been identified on the image, all other points and lines can be determined (because the object and the image are congruent).
- If the centre of rotation is a point on the shape, then that point does not move under the transformation.

Students could use their understanding of rotation be conjecture what would happen if the centre of rotation was a point inside the shape.

Example 3: *Four transformations of the blue object are shown here.*



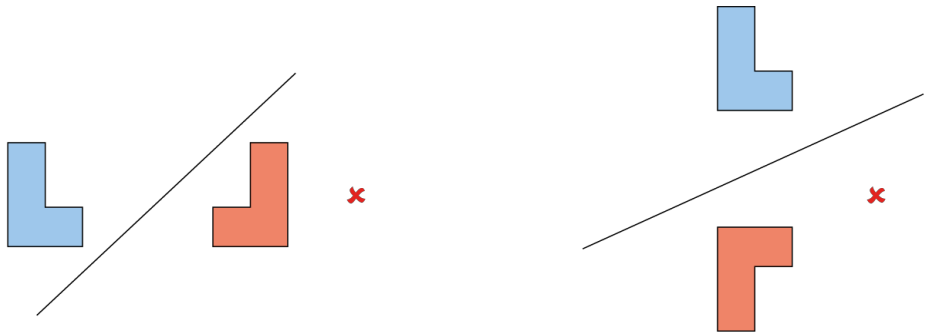
a) *Which image is not a rotation of the object?*
b) *Fully describe the transformation of the object to each image.*

The first part of *Example 3* draws students' attention to what a rotation *is not* by the inclusion of image 4, which is a reflection of the object in the line DB.

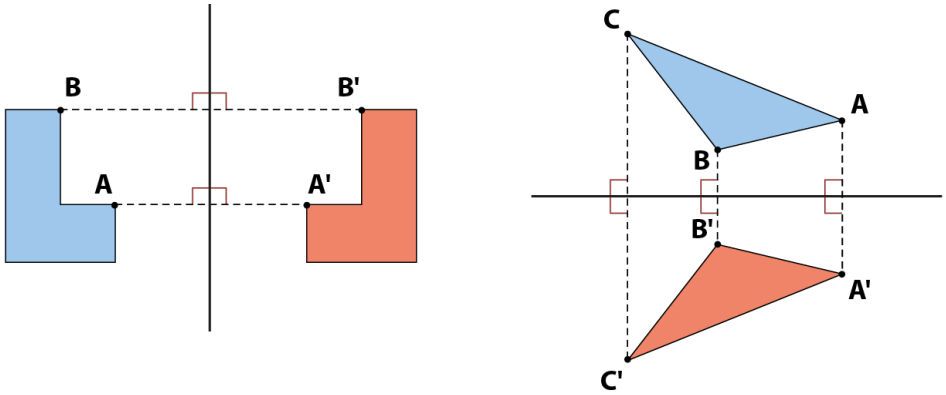
Precise language to describe a rotation can be an effective tool in drawing students' attention to the necessary features of rotation. For example: *'Object A is rotated 90 degrees anti-clockwise around the point B.'*

Understand the minimum information required to describe a reflection (line of reflection)

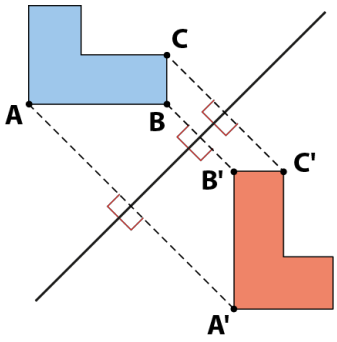
Common difficulties and misconceptions: many students are intuitively able to reflect an object in a vertical or horizontal mirror line, but the reflection of an object in a line that is not vertical or horizontal often proves a challenge. A common misconception is that the image will remain in the same vertical or horizontal plane:



This misconception may well be due to an overuse of vertical or horizontal lines of reflection. It is important for students experience a wide range of non-standard, as well as standard, examples of lines of reflection. Explanations which draw students' attention to the fact that every line which joins a point on the object to its image is **perpendicular** to the line of reflection, will support students in understanding the relationship between the object/image and line of reflection.

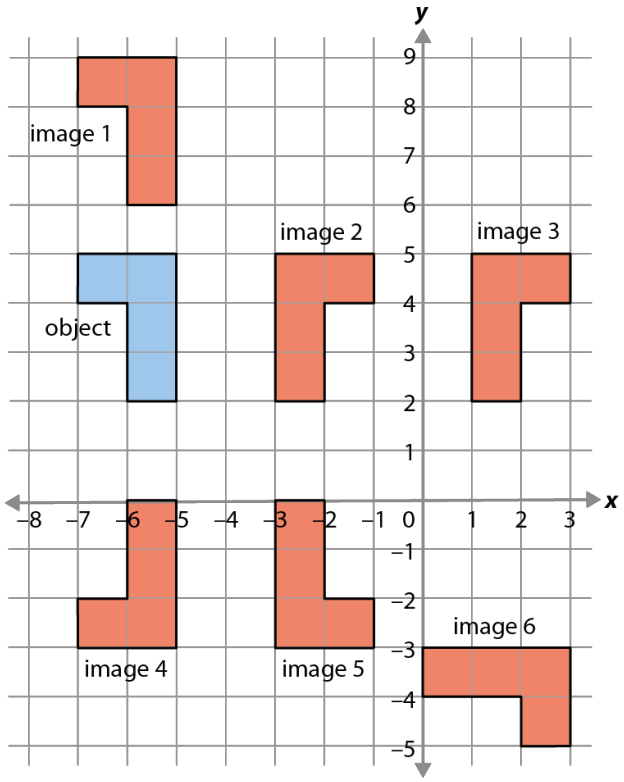


Making this clear when working with vertical and horizontal lines of reflection will support students in generalising the idea and help to avoid such misconceptions.



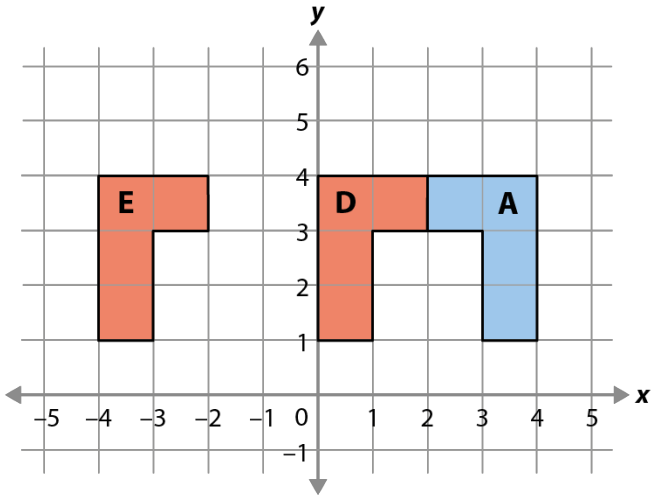
As well, using examples that include objects and images in any quadrant of the graph and overlapping the axes. Mirror lines should include the axes and lines with a gradient of 1 and –1, including $y = x$ and $y = -x$. Examples are given below.

Example 1: Which images are reflections of the object?



Example 1 shows transformations of the object in different positions and orientations. Images 2, 3, 4 and 6 are reflections. Image 5 is a rotation chosen as it is often confused as being a reflection in the line $y = x$. Image 1 has been chosen to draw attention to the change in 'sense' that is a necessary feature of a reflection.

Example 2: *Describe the transformations of shape A onto shape D and shape E.*

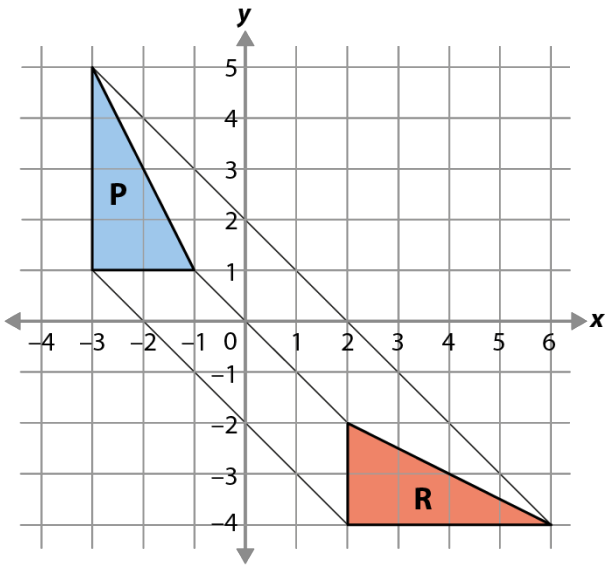


In *Example 2*, image D is in contact with object A, highlighting that there does not need to be a space between the object (and image) and the mirror line. Image E is a reflection in the *y*-axis.

There is an opportunity here for students to describe the axis of reflection of A onto E as either 'the *y*-axis' or '$x = 0$'.

Students could also consider whether objects and their images can overlap when reflected. By imagining the change in the mirror line from E to D as a dynamic change, students may be able to visualise the impact of moving the line and so consider the impact of moving the line even further right than that for image D.

Example 3: *Object P has been reflected to give the image R. The vertices of P and R have been joined.*



The examples used so far have been aligned in such a way that the angle of the mirror line has not been an explicit consideration. By changing the angle of the mirror line, greater depth of understanding about the angle between the object and line can be revealed.

Recognising the lines joining equivalent points on the object and image are parallel, combined with their understanding so far, students should begin to recognise that the mirror line is **always** perpendicular to the lines joining the equivalent vertices.