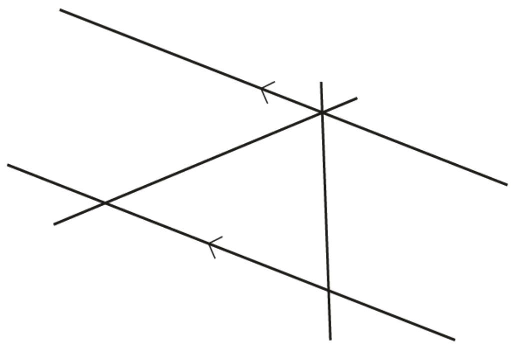
Geometrical properties: polygons

Overview

Students will have had opportunities to develop their spatial awareness and geometrical intuition in Key Stage 2 through situations involving angles (angles meeting at a point, angles on a straight line, vertically opposite angles and angles in regular polygons) and similar shapes. They will be aware of the geometrical facts and properties inherent in these situations. An important development throughout Key Stage 3 is to be able to reason and construct proofs for why such facts and properties hold and begin to understand the nature of mathematical proof.

Teaching and learning geometry offers an opportunity for students to think about relationships and structures, reason with them, prove results and distinguish proof from demonstration.

In the context of angles, the geometry of intersecting lines and the connections and deductions that can be made from diagrams, such as this:



provide rich opportunities in which students can be encouraged to build logical, deductive arguments. Students develop a narrative, connecting and combining known facts in order to generate further mathematical truths. The order of teaching needs careful consideration as some proofs of the angle sum of a triangle rely on an understanding of the angles generated when a transversal crosses a pair of parallel lines.

Prior learning

Before beginning geometrical properties of polygons at Key Stage 3, students should already have a secure understanding of the following learning outcomes from study at upper Key Stage 2:

- Recognise angles where they meet at a point, are on a straight line, or are vertically opposite, and find missing angles.
- Compare and classify geometric shapes based on their properties and sizes and find unknown angles in any triangles, quadrilaterals, and regular polygons.

The NCETM has created the following Key Stage 2 ready-to-progress criteria to support teachers in making judgements about students' understanding and knowledge.

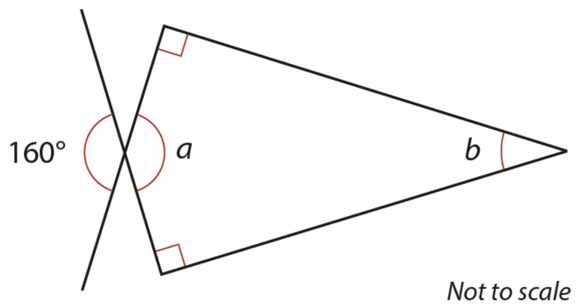
4G-2 Identify regular polygons, including equilateral triangles and squares, as those in which the side-lengths are equal and the angles are equal. Find the perimeter of regular and irregular polygons.

5G-1 Compare angles, estimate and measure angles in degrees (°) and draw angles of a given size.

6G-1 Draw, compose, and decompose shapes according to given properties, including dimensions, angles and area, and solve related problems.

Checking prior learning

The following activities from the Standards & Testing Agency past mathematics papers offer useful ideas for teachers to use to check whether prior learning is secure.

Reference	Activity
2016 Key Stage 2 mathematics paper 2: reasoning Question 17	*Calculate the size of angles a and b in this diagram.* 

2018 Key Stage 2 mathematics paper 2: reasoning Question 14	*Two of the angles in a triangle are 70° and 40°.* *Jack says,*  *Explain why Jack is not correct.*

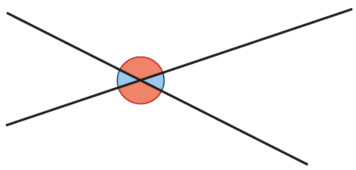
Language

alternate angles, congruent (figures), corresponding angles, interior angle, supplementary angle, transversal

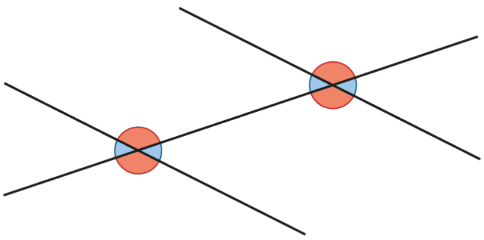
Progression through key ideas

Understand and use angle properties

In Key Stage 2, students should have studied that vertically opposite angles are equal:

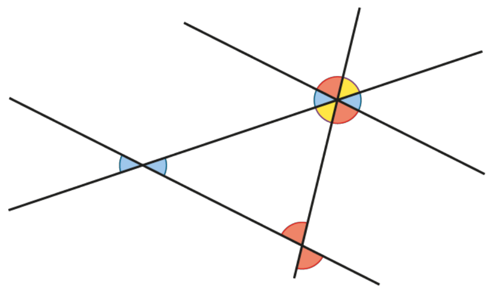


In Key Stage 3, students will use this fact to deduce that a translation of one of the lines will create a second, equivalent pair of vertically opposite angles:



They will then be able to identify pairs of equal angles (some of which are named alternate or corresponding) and other relationships, such as pairs of supplementary angles (i.e. with a sum of 180°).

Throughout Key Stage 3, students are encouraged to use what they know to construct a logical argument and deduce other facts. By offering carefully-selected contexts that encourage reasoning, students can construct and understand proofs, such as the angles in a triangle always sum to 180°.



Key ideas

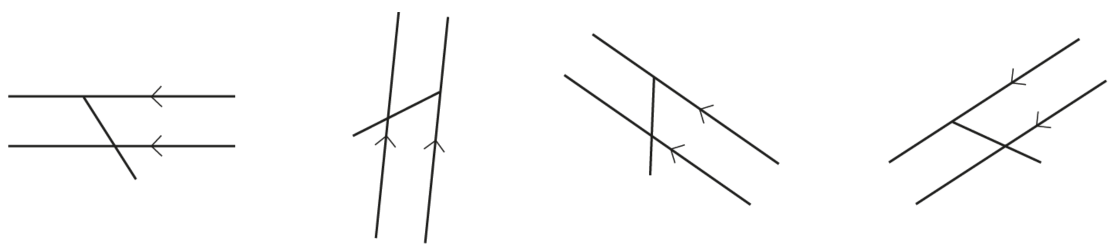
- Understand that a pair of parallel lines traversed by a straight line produces sets of equal and supplementary angles*
- Know and understand proofs that in a triangle, the sum of interior angles is 180 degrees*
- Know and understand proofs for finding the interior and exterior angle of any regular polygon
- Solve problems that require use of a combination of angle facts to identify values of missing angles, providing explanations of reasoning and logic used

Exemplified significant key ideas

Understand that a pair of parallel lines traversed by a straight line produces sets of equal and supplementary angles

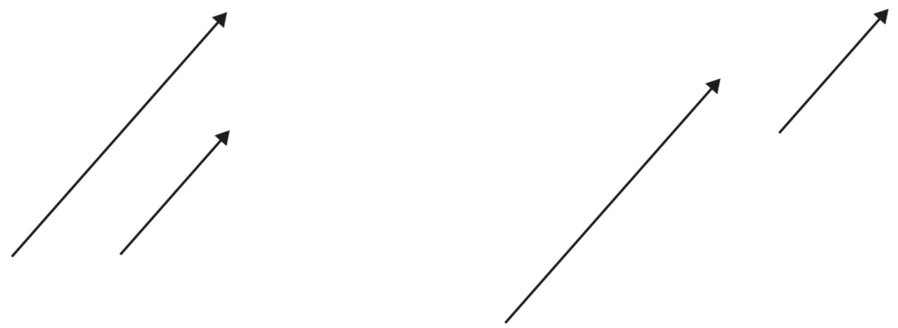
Common difficulties and misconceptions: students often confuse alternate and corresponding angles and, consequently, may not be able to identify examples of these in any but the simplest of diagrams and shapes.

They should be given plenty of opportunities to notice such angles in non-standard diagrams, such as these:



They should be encouraged to use reasoning based on the inherent geometrical structure rather than memorising standard diagrams and using phrases such as ‘F’ and ‘Z’ angles, which do not support reasoning and mathematical thinking.

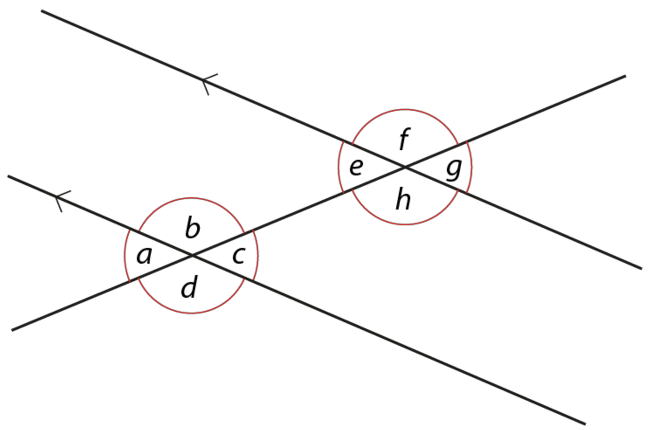
Another common difficulty is that students recognise and accept lines as being parallel only if they are of a similar length and position; off-set lines are often not perceived as being parallel, particularly when there is little overlap, as below:



It is worth encouraging the use of precise language when describing angle properties. For example, using ‘alternate angle’ rather than ‘Z angle’; and stating the full angle property when reasoning, for example, *‘x = 45° because alternate angles are equal.’*

Examples are given below.

Example 1*: Write as many equations (or statements) as you can to show the relationships between the angles in this diagram.*



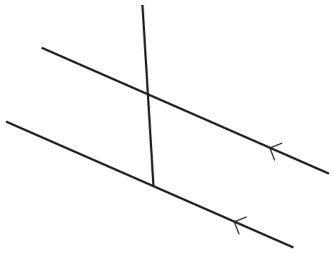
Explain your reasoning.

Example 1 offers an opportunity for students to generate statements about relationships between angles created on parallel lines. They should be encouraged to fully explain their reasons, using precise language.

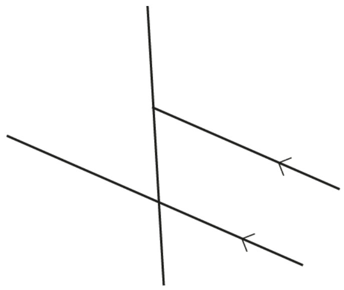
With this example, tracing paper can be used to show how the cluster of four angles *a*, *b*, *c* and *d* can be translated along the transversal to fit exactly on top of the *e*, *f*, *g*, *h* cluster, in order to support students’ understanding.

Example 2: *If the acute angle in each of these diagrams is 50°, what are the other angles?*

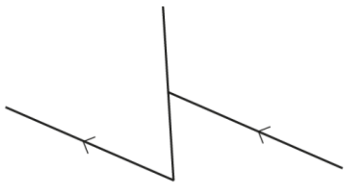
a)



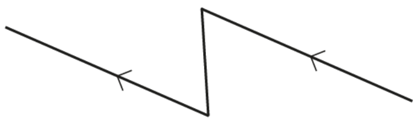
b)



c)



d)

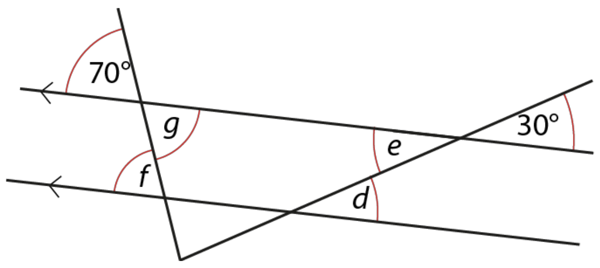


Example 2 offers students some non-standard examples to explore. By systematically making small changes to a diagram, students will be able to see examples of equal and supplementary pairs of angles within the overall structure of two incomplete sets.

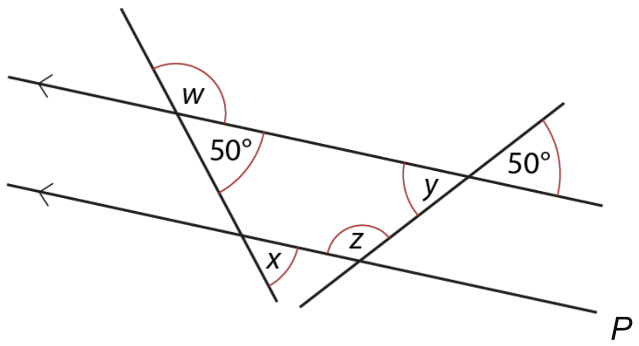
Attention can also be drawn to the fact that the reflex angles in parts *c* and *d* are made up of a combination of one of the acute and two obtuse angles. Students could be challenged to draw a similar diagram where the reflex angle is made up of one obtuse and two acute angles.

Example 3: *Find the value of the labelled angles.*

a)



b)

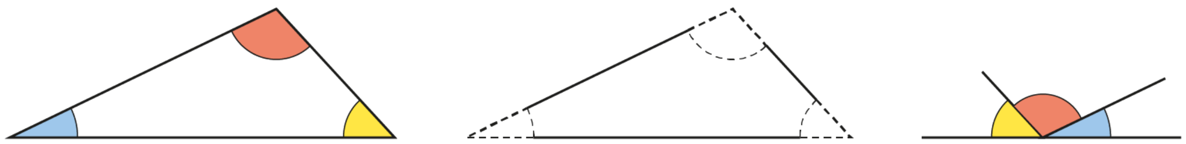


Example 3 has been designed for students to use and apply their knowledge and skills of identifying corresponding, alternate, vertically opposite and supplementary angles, rather than using other properties such as the sum of interior angles in a triangle being 180°.

Whole-class discussion based around students' responses to this example may be deepened by considering other properties. For example, in part *a*, attention could be drawn to the acute angle that is neither 30° nor 70° and students could discuss how this might be found. Similarly, in part *b*, attention could be drawn to the isosceles trapezium and deductions made about other angles, using notions of symmetry and equal length.

Know and understand proofs that in a triangle, the sum of interior angles is 180 degrees

Common difficulties and misconceptions: students often confuse a demonstration for a proof. In Key Stage 2, students learnt that the sum of the interior angles of a triangle is 180°, and will have used this fact to calculate missing angles. They may have seen a demonstration of this fact involving cutting out a paper triangle, tearing off the three corners and showing that the angles can be placed together on a straight line:

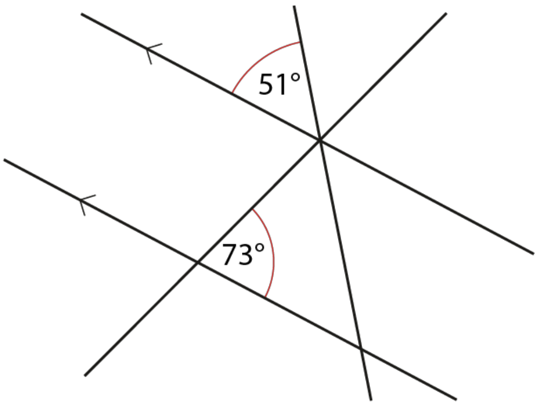


It is important that students appreciate that, while this indicates that such a relationship might be true (and indeed is a very useful way of appealing to students' intuition), it is a demonstration and not a proof.

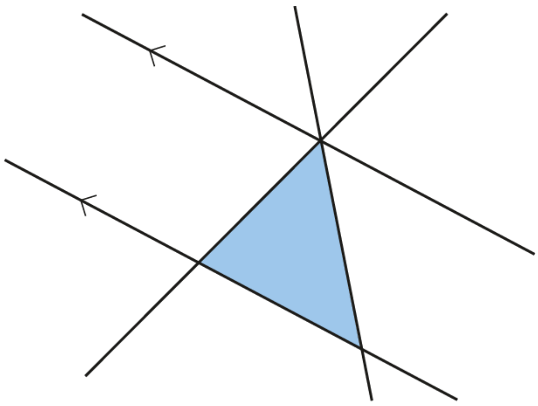
In Key Stage 3, students will develop their understanding of what is meant by mathematical proof. This is likely to include understanding proof as a form of convincing argument based on logical deduction and an expression of generalisation, as opposed to checking against a few specific cases. Students are also developing an understanding about the conventions of communicating proof, including the use of language such as 'if … then', 'therefore' and 'because', and correct and unambiguous use of mathematical symbolism. Examples are given below.

Example 1:

a) Fill in all the missing angles. Give a clear explanation for each answer.



b) What do you notice about the three angles in the shaded triangle?

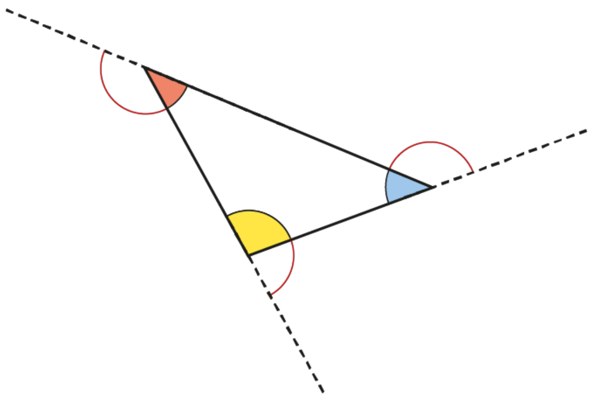


c) Where else can you see these three angles next to each other?

In *Example 1*, a triangle is created using a pair of parallel lines and two transversals. In part *a*, students should be encouraged to use angle facts with which they are familiar to find as many missing angles as possible. This build-up of facts, deduced from already known and understood relationships, allows students to arrive at a convincing argument about the sum of the angles in a triangle for themselves.

Part *c* draws attention to the relevance of 180° as half a full turn or the sum of the angles on a straight line.

Example 2: *Given that the exterior angles of a polygon add up to 360°, prove that the interior angles of this triangle add up to 180°.*



Example 2 gives students an opportunity to explore an alternative proof for the angle sum of a triangle, this time connecting to their knowledge of exterior angles.

It may be helpful to begin with students drawing a triangle on the floor of the classroom (maybe on a large piece of sugar paper). Demonstrate walking around the perimeter, taking note of the direction faced at the start; the turns made at each vertex; and the direction faced upon return to the starting point.

An alternative is to use a version of this diagram on a computer or interactive whiteboard and demonstrate how it can be scaled down proportionately to show how the triangle reduces to a point and the external angles clearly show a full turn.

Example 3*: Emma and Samira each show that the angles in a triangle add up to 180°.*

Emma constructs a triangle using a pair of compasses and a ruler, measures each of the interior angles and adds them up. They have a sum of 180°. She repeats this for two different triangles and finds the same result.

Samira cuts out a paper triangle, tears off all three corners and places them along the edge of a ruler to show that they fit together and lie on a straight line.

Give reasons why neither Emma nor Samira has produced a convincing argument.

Example 3 offers an explicit opportunity to discuss why demonstrations, such as these, are not convincing proofs Consider asking students to repeat Emma and Samira's approaches. Through doing the activity themselves, students will come to appreciate the role of measurement and estimation in both methods.

Ask questions which might promote deeper thinking from students, such as '*How do we know that the sum of the angles is not 179° or 181°?*' '*Do you think it is a coincidence that the angle sum of a triangle is exactly half a turn? Why do you think that might be?*'

Constructions

Overview

In Key Stage 2, students will have learnt about the properties of certain geometric shapes and used these properties to compare and classify shapes. They will also have had experience of drawing certain shapes using a ruler and angle measurer. Developing this work in Key Stage 3, students will learn the ruler and compass constructions of:

- triangles of given lengths
- a perpendicular bisector of a line segment
- a perpendicular to a given line through a given point
- an angle bisector.

An important awareness is that these constructions are based on the geometrical properties of a few key shapes (a circle, an isosceles triangle and a rhombus). A deep understanding and awareness of these geometrical properties will support students in gaining a conceptual overview of these constructions and guard against constructions being learnt mechanically as a set of procedural steps.

Prior learning

Before beginning constructions at Key Stage 3, students should already have a secure understanding of the following learning outcomes from study at upper Key Stage 2:

- Draw 2D shapes using given dimensions and angles.
- Compare and classify geometric shapes based on their properties and sizes and find unknown angles in any triangles, quadrilaterals, and regular polygons.

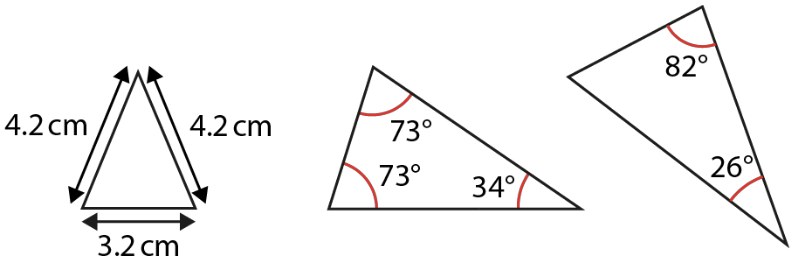
The NCETM has created the following Key Stage 2 ready-to-progress criteria to support teachers in making judgements about students' understanding and knowledge.

5G-1 Compare angles, estimate and measure angles in degrees (°) and draw angles of a given size.

6G-1 Draw, compose, and decompose shapes according to given properties, including dimensions, angles and area, and solve related problems.

Checking prior learning

The following activities from the NCETM primary assessment materials offer useful ideas for teachers to use to check whether prior learning is secure.

Reference	Activity
Year 6 page 35	*Which of these triangles are isosceles?* *Explain your decisions.* 
Year 6 page 35	*Accurately draw two right-angled triangles with sides of different lengths.* *Compare them and describe what's the same and what's different about them.*

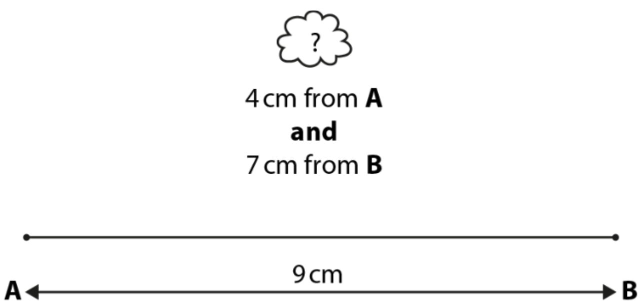
Language

altitude of a triangle, arc, bisector, congruent, construction, line, line segment, locus, perpendicular

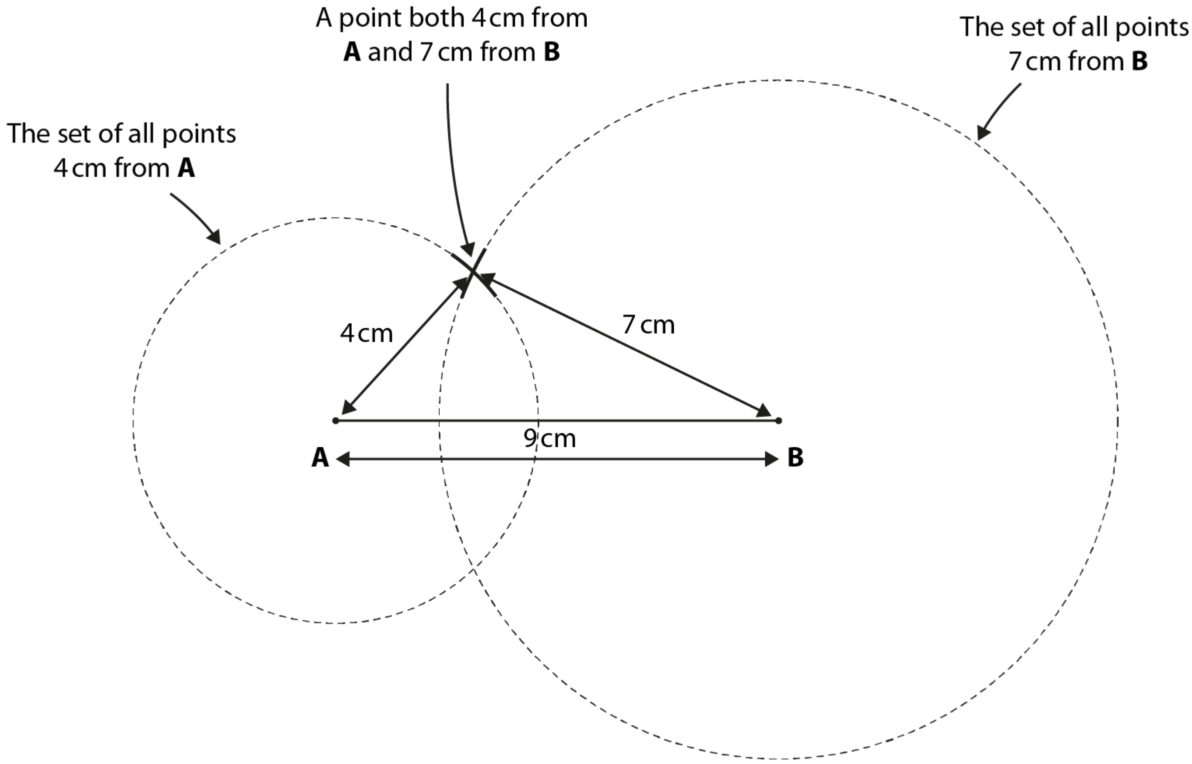
Progression through key ideas

Use the properties of a circle in constructions

When faced with the problem of constructing a triangle with lengths of, for example, 4 cm, 7 cm and 9 cm, students' intuition may be to do so using a ruler. Trying to solve this problem using a ruler will be a useful exercise, as it draws attention to the challenge of finding a point that is a specified distance from one point and, simultaneously, a specified distance from another, as shown:



A key awareness is that drawing a circle creates an infinite set of points, all of which are equidistant from its centre. Students will need plenty of experience of using a ruler and a pair of compasses to appreciate the nature of the construction. They should also become aware that drawing of carefully-placed arcs is more efficient than drawing full circles.



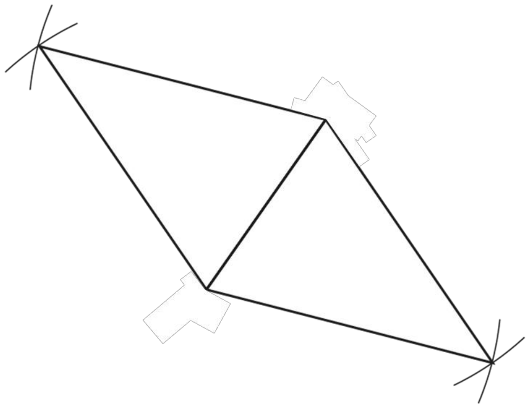
Once students can construct a scalene triangle with ease and efficiency, they can be challenged to construct other shapes (for example, equilateral and isosceles triangles and rhombuses). This will not only provide opportunities for students to become fluent with the construction processes, but also, importantly, to engage in some early discussions about the basic properties of these shapes. These discussions can then be extended when focusing on the ruler and compass constructions.

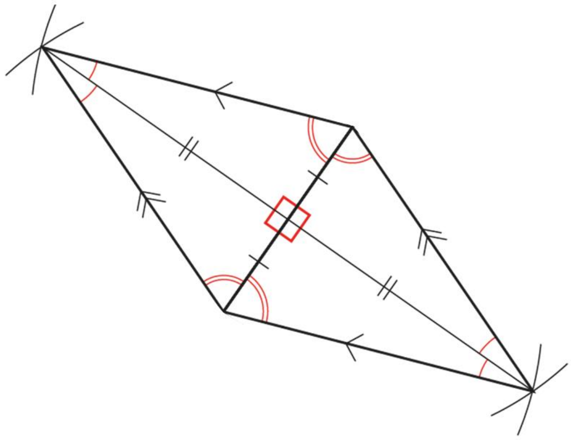
Key ideas

- Understand a circle as the locus of a point equidistant from a fixed point
- Use intersecting circles to construct triangles and rhombuses from given lengths

Use the properties of a rhombus in constructions

Using their previous understanding of how to use arcs of circles to construct isosceles triangles and rhombuses, students can explore more closely the properties of these shapes. For any isosceles triangle, the altitude of the triangle bisects the base at right angles and bisects the angle at the vertex. A rhombus comprises two congruent isosceles triangles joined at a common edge and therefore the diagonals are perpendicular bisectors which bisect their associated internal angles.





The key awareness is that when a rhombus is constructed, other geometric properties are created and these are utilised in the standard constructions. Students should be encouraged to identify these properties and to locate the rhombus in the standard constructions.

Key ideas

- Be aware that the diagonals of a rhombus bisect one another at right angles
- Be aware that the diagonals of a rhombus bisect the angles
- Use the properties of a rhombus to construct a perpendicular bisector of a line segment*
- Use the properties of a rhombus to construct a perpendicular to a given line through a given point
- Use the properties of a rhombus to construct an angle bisector

Exemplified significant key ideas

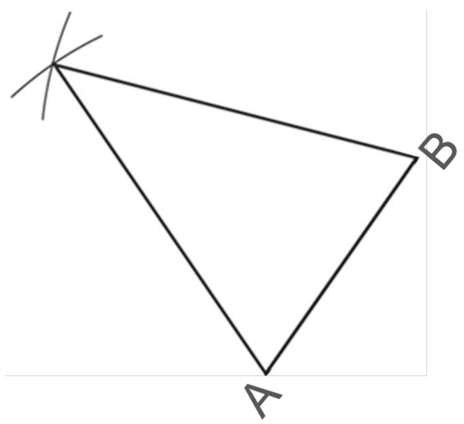
Use the properties of a rhombus to construct a perpendicular bisector of a line segment

Common difficulties and misconceptions: some students may experience difficulties surrounding the mathematical language used in this key idea. They should understand the difference between drawing a figure by eye, and producing an accurate construction based on the geometrical properties of the figure. Previously students will have drawn a perpendicular bisector by using a ruler to determine the midpoint of a line and a protractor to judge a right angle. In a construction, it is geometrical properties, not measurement, which are used to produce the required result.

Students should be given time to practise using construction equipment accurately. It is likely that they will not have used a pair of compasses frequently (if at all) during Key Stage 2, and so students often, initially, lack coordination. They may need support in setting up their equipment and should be encouraged to check their working, with the aim to be within ±2 mm and ±2° of the required measurements.

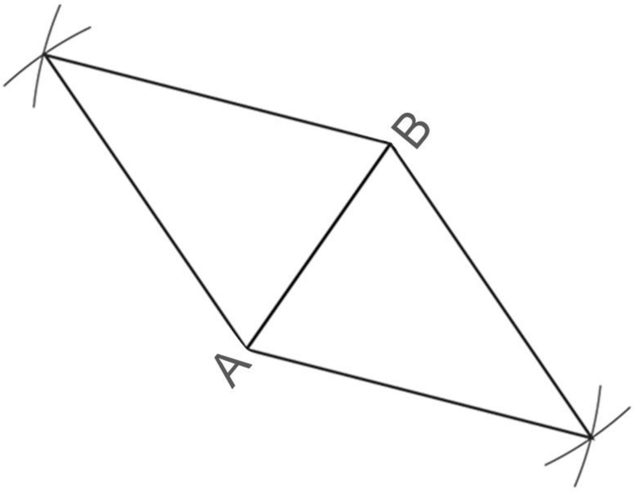
Students can find it difficult to memorise the various steps in creating constructions when they do not link this work to other knowledge about geometrical properties. They will be helped considerably if they are aware that constructing a perpendicular bisector of a line segment is not an isolated concept but linked to the properties of circles and rhombuses. Examples are given below.

Example 1*: This diagram is a construction of an isosceles triangle.*



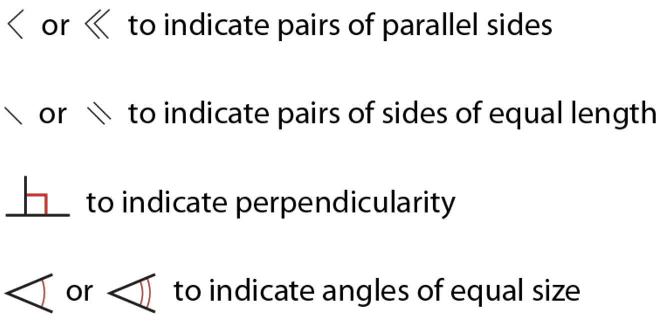
a) *Write down (or indicate on the diagram) as many properties of this triangle as you can.*
b) *Draw in an altitude of this triangle. Are there any other properties that you can now state?*

Add a congruent isosceles triangle to the diagram, like this:



c) *What shape have you made?*
d) *Write down (or indicate on the diagram) as many properties of this shape as you can.*
e) *Draw in the longest diagonal of this shape. Are there any other properties that you can now state?*

Example 1 offers students an opportunity to connect their knowledge of constructing triangles with properties of a rhombus. Students will need to be familiar with technical language such as 'altitude' and 'diagonal' to complete this example. Encourage students to use such symbols as:



as well as written descriptions when completing their diagrams.

Once students have constructed this, it is important to discuss the relationships embedded in the diagram, offering prompts such as:

- *'Can you see two lines that bisect each other at right angles?'*
- *'Can you see an angle that has been bisected?'*
- *'And another, and another, …?'*

For a deep and connected understanding of the ruler and compass constructions, students could imagine a rhombus and construct the part of it that will produce the required construction. For example, if required to construct a perpendicular bisector to the line AB, encourage students to imagine the line AB as being a diagonal of a rhombus and then proceed to draw the rhombus (or part thereof).

Example 2: *The line segment AB is 8 cm long.*

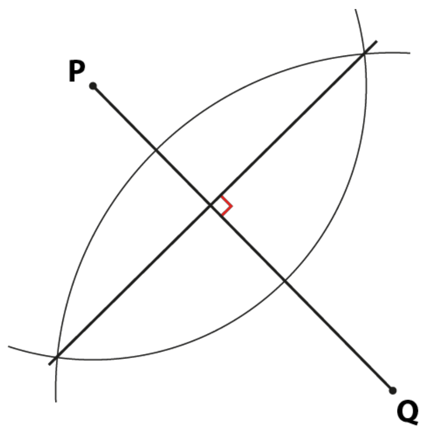
a) Which of these methods could be used to construct the perpendicular bisector of AB?
   (i) Constructing arcs which are 3 cm from both A and B.
   (ii) Constructing arcs which are 4 cm from both A and B.
   (iii) Constructing arcs which are 5 cm from both A and B.
   (iv) Constructing arcs which are 8 cm from both A and B.

b) Which is the best method? Explain your reasoning.

In *Example 2*, students must decide which of the four options could result in the construction of the perpendicular bisector of the line segment AB. The size of the arcs has been carefully chosen to prompt students to consider which would intersect both above and below the line segment AB and allow an accurate construction of the perpendicular bisector. In part *i*, arcs of 3 cm would not intersect. In part *ii*, the arcs of 4 cm would only bisect the line segment at the midpoint. Parts *iii* and *iv* could both be used, but arcs of 8 cm, as in part *iv*, are unnecessarily large.

In this task, students should be initially encouraged to consider the construction conceptually, but some may then need to construct them to fully understand which will work.

Example 3: *Tinashe has attempted to construct the perpendicular bisector of the line segment PQ.*



Comment on his construction.

a) Which quadrilateral has he constructed?
b) Why might this have happened?
c) What do the properties of this shape indicate about the diagonals?

Example 3 shows an incorrect construction. At first glance it may look like the perpendicular bisector has been constructed, but students should identify that the arcs drawn do not have equal radius and so a kite has been constructed instead of a rhombus. The diagonals of a kite are perpendicular but do not bisect each other, so a perpendicular bisector has not been constructed. This reinforces the need to keep the pair of compasses open to the same length in order to construct a rhombus.

Using non-standard examples of constructions where the line to be bisected is not horizontal or vertical helps students to discern essential and non-essential features.