converting both to sixths means that both have the same unit and addition is relatively straightforward.

Students should develop an understanding of the additive structures underpinning the operations, as well as fluency with strategies for adding and subtracting a wide range of types of fractions (including improper fractions).

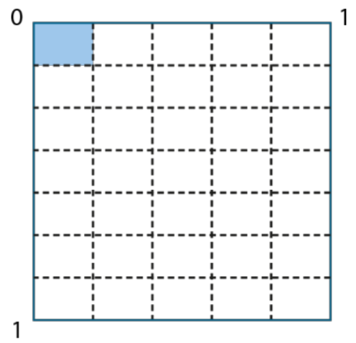
Key ideas

- Understand the mathematical structures that underpin the addition and subtraction of fractions
- Generalise and fluently use addition and subtraction strategies to calculate with fractions and mixed numbers

Know, understand and use fluently a range of calculation strategies for multiplication and division of fractions

Students having an unconnected view of the curriculum can result in an entirely instrumental and procedural approach to mathematics, with no sense of conceptual coherence. It is, therefore, important that students see fractions or rational numbers as a part of a unified number system and that the operations on such numbers are related and connected to previously taught and learnt concepts for integers. For instance, the area model used for multiplication with integers can also be used for fractions.

For example, $\frac{1}{7} \times \frac{1}{5}$ can be represented as:



$\frac{1}{7} \times \frac{1}{5}$

In the key ideas below, multiplication with mixed numbers has been given a separate focus. The rationale for this is that the different possible representations for multiplying mixed numbers – such as converting to improper fractions or partitioning as a pair of binomials, e.g. $(2 + \frac{3}{4})(1+ \frac{2}{3})$ – may offer different and deeper insights into multiplication.

Key ideas

- Understand the mathematical structures that underpin the multiplication of fractions*
- Understand how to multiply unit, non-unit and improper fractions*

- Generalise and fluently use strategies to multiply with mixed numbers (e.g. $2\frac{3}{4} \times 1\frac{2}{3}$)
- Understand the mathematical structures that underpin the division of fractions
- Divide a fraction by a whole number
- Divide a whole number by a fraction
- Divide a fraction by a fraction

Exemplified significant key Ideas

Understand that a fraction represents a division and that performing that division results in an equivalent decimal

Many students, when picturing a fraction such as three-fifths, will imagine a single object split into five equal parts, with three of them selected. This image of a fraction is encouraged by examples such as *'A cake is cut into five equal parts. Three children each take one part. How much of the cake is eaten?'* In this interpretation both the numerator and denominator of the fraction represent the same 'unit' (in this case, slices of a cake).

While this image is accurate, it is incomplete, and a second understanding of fractions (the quotient construct) should also be considered. In this interpretation of the fraction notation, the numerator and denominator represent different units. For example, in the question *'Three cakes are shared equally between five children. How much cake does each child eat?',* the numerator represents the number of cakes while the denominator represents the number of children.

Nunes and Bryant (2009) state that most children are introduced to fractions through the part–whole model and have less experience of fractions as a quotient. They also suggest that, although the differences between these models are subtle, it is a crucial distinction with at least four implications for students' understanding of fractions. Here is a summary:

- The understanding of improper fractions may be easier when using the quotient construct: five cakes shared between three children makes is an easier way to understand five-thirds than one child eating five-thirds of a cake.
- Students understand that the way in which a quantity is partitioned doesn't matter, as long as the sharing is equal. For example, if five cakes are to be shared among three children then each cake doesn't **have** to be cut into five equal parts, with each child getting three of them.
- It is suggested that ordering fractions may be easier when using the quotient construct. It is likely to be easier to reason about which is larger, three-fifths or three-sixths, when considered as three cakes shared between five people or three cakes being shared between six people.
- Understanding the equivalence of fractions may also be easier using the quotient construct as students may be able to reason that doubling the number of cakes and the number of children won't affect the amount of cake any child eats.

A challenge students might face when considering fractions as division is that the fraction is both the process needed to calculate the answer and the answer itself, i.e. when sharing three cakes between five people, the calculation used is 3 ÷ 5 and the answer is three-fifths.

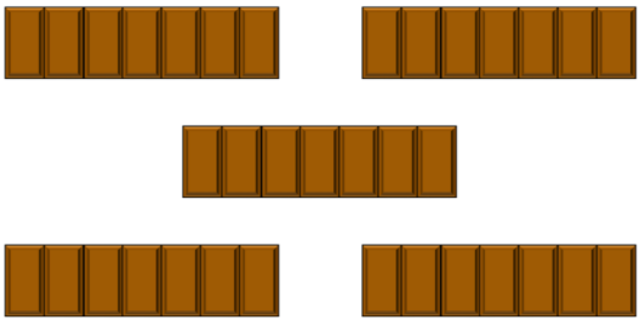
In addition, students who may not fully understand decimal and fraction notation often try to convert fractions to decimals by replacing the fraction bar with a decimal point for example, writing five-thirds as 5.3. The use of representations such as shading a hundred square may help to overcome this misconception.

Example 1: *A chocolate bar has seven equal sections like this.*

Mark eats five sections of the bar.

a) *Shade the chocolate bar to show this.*
b) *What fraction of the bar has Mark eaten?*

Five chocolate bars are shared equally between seven children.



c) *Shade the diagram to show how this sharing might happen.*
d) *What fraction of a bar does each child get?*

In *Example 1,* the representation allows an opportunity to explore the difference between partitioning and finding the quotient.

In the second part, students might be asked to find different ways to share the bars (for example, how can the bars be shared with the fewest ‘breaks’ of the chocolate) and this would allow for more reasoning around the connection between division and the resulting fraction of $\frac{5}{7}$.

Example 2:

a) *Mark the number 3 on this number line.*

b) *Mark the number $\frac{3}{7}$ on this number line.*

c) *What's the same and what's different about parts a and b?*

In *Example 2*, the number line representation is used to give a context for two different ways to interpret a fraction.

Although placing a 3 on the 0 to 7 number line, and identifying $\frac{3}{7}$ on the 0 to 1 number line are mathematically the same (they are both $\frac{3}{7}$ of the way along the identified section of the line), the students may interpret part a as connected with division, as they are sharing the 7 into seven equal parts in order to identify where 3 goes, and part b as being about fractions. This task is intended to give a context to make the common language and structure of division and fractions explicit.

For both *Examples 1* and *2,* it is important that the students have an opportunity to verbalise and discuss their thinking, sharing their understanding and hearing alternative perspectives on each situation, so they become aware of different ways of visualising and interpreting contexts that are mathematically identical.

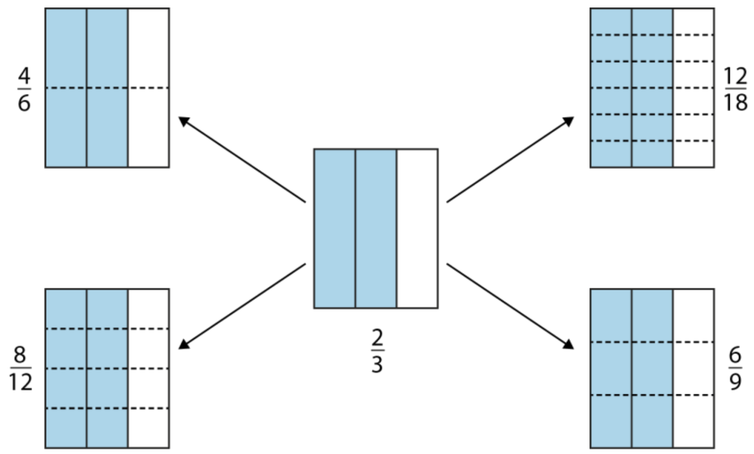
Understand the process of simplifying fractions through dividing both numerator and denominator by common factors

Common difficulties and misconceptions*:* when simplifying, or 'cancelling', fractions, students may know to look for a common factor by which to divide both the numerator and denominator. However, if this has been learnt as a procedure, with no understanding of the reason behind it, a number of misconceptions may arise. Students may, for example, think that since a division has occurred, the size of the fraction has changed. Students may also think that a 'cancelled down' fraction can be obtained by subtracting from both the numerator and the denominator, rather than by dividing.

Students should understand that the process of cancelling is the inverse of the process for obtaining equivalent fractions. It involves the scaling down of both the numerator and the denominator and, therefore, maintaining the same (multiplicative) relationship between them.

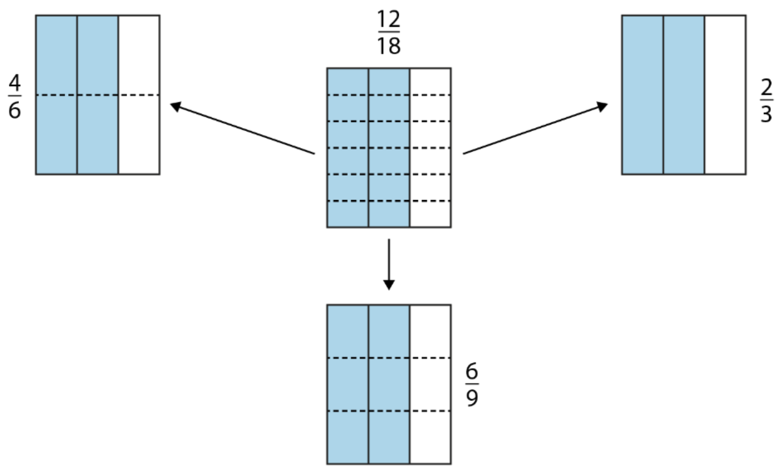
The use of diagrams that reveal the structure of the mathematics will be important to support students' deep understanding of the concept and develop their fluency with the procedure.

For example, by sub-dividing we can create fractions equivalent to $\frac{2}{3}$:



Note what has happened to the number of blue sections and the number of white sections each time, and how this translates into the change of numerator and denominator.

By removing some sub-divisions, we can create fractions equivalent to $\frac{12}{18}$:



Again, note what has happened to the number of blue sections and the number of white sections each time.

Diagrams such as the ones above will help students to appreciate that, as the denominator is halved (or divided by three, four, five, etc.) then the numerator must also be halved (or divided by three, four, five, etc.) to retain equality. However, it is important not to create the misconception that one always has to multiply by two, then by three, then by four, etc. Students need to appreciate that multiplying the numerator and denominator by the same integer yields a fraction equivalent to the original. Conversely, dividing the numerator and the denominator by the same common factor obtains an equivalent fraction.

As students work more symbolically, they may recognise common factors such as two, five and ten within fractions but may not check rigorously enough to arrive at a fraction in its simplest form. Students may simplify $\frac{12}{42}$ to $\frac{6}{21}$ but not cancel by three to arrive at $\frac{2}{7}$

Students will have been introduced to fractions in several ways, so illustrating fractions as representations is important. When students' experience of fractions is solely through the symbolic representation, then the language of 'two over three' can dominate. This does not support students' understanding. Using diagrams, such as the ones above, can support and encourage use of the term 'out of', as in 'two out of three'.

Example 1: *Two lines on this multiplication grid have been highlighted:*

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80

a) *Discuss the patterns you see in the numbers on these two highlighted lines.*
b) *What do you notice about the two sets of numbers; how are they related?*
c) *What can you say about the fractions that are formed by taking a number on the upper highlighted line as the numerator and the number below it on the lower highlighted line as the denominator?*
d) *Highlight two different lines on the multiplication grid. What fractions are revealed and what do you know about them?*

In the multiplication grid in *Example 1*, $\frac{3}{5}$ is equivalent to $\frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \frac{15}{25}$.

Students' attention can be drawn to the fact that, in each case, the numerator is three-fifths of the denominator. Use of the grid in this way will support students in seeing, for example, $\frac{18}{30}$ as $\frac{3\times6}{5\times6}$, and therefore equivalent to $\frac{3}{5}$.

Where appropriate, draw students' attention to non-integer examples, such as $\frac{2.5}{10}$. Students should recognise that these can be expressed as proper fractions through multiplying by a convenient number. Note, multiplying a terminating decimal by ten or a power of ten will always produce integers, but not necessarily a fraction in its simplest form, for example $\frac{12.5}{20} = \frac{125}{200}$.

Example 2: *Is each fraction in its simplest form? Explain your reasoning.*

a)	$\frac{8}{18}$	*e)*	$\frac{12}{18}$
b)	$\frac{9}{18}$	*f)*	$\frac{13}{18}$
c)	$\frac{10}{18}$	*g)*	$\frac{14}{18}$
d)	$\frac{11}{18}$	*h)*	$\frac{15}{18}$

Once students are fluent with generating equivalent fractions by multiplying both numerator and denominator by any integer, their attention can be shifted to noticing whether any common factors can be removed from certain fractions, as in *Example 2*. Notice how the denominator is kept constant so that students can more easily focus on the idea of a common factor without being distracted by having to find all the factors of a different denominator each time.

It will be important for students to articulate how they know when a fraction is in its simplest form. Questioning should encourage the use of some standard language, such as, *'… because the numerator and the denominator have no common factors (other than one)'*.

Example 3: *Explain why 8 ÷ 2, 80 ÷ 20 and 800 ÷ 200 all give the same result.*

A significant idea is to conceptualise fractions as a division, $\frac{3}{4} = 3 \div 4$. This is not always grasped in the study of fractions at Key Stage 2, and is crucial to developments at Key Stage 3. Working with integers offers a more familiar context to begin this work. Students could be asked to give several explanations (including drawing diagrams) for why the answers to *Example 3* are all four.

Understand the mathematical structures that underpin the multiplication of fractions

Common difficulties and misconceptions: when multiplying fractions, students' awareness can be directed to the idea (possibly made explicit for the first time) that the product of two numbers can be smaller than either of those numbers. Students who see multiplication only as repeated addition (and, therefore, as always making something bigger) will find this difficult.

For students to make sense of this, they need to deepen their understanding of multiplication to include scaling and the idea that multiplying any number by a fraction between one and zero makes that number smaller.

For example, when considering a calculation such as $3 \times \frac{3}{4}$, it will be important to encourage students to think of this in two ways:

- the number $\frac{3}{4}$ taken three times, i.e. $\frac{3}{4} + \frac{3}{4} + \frac{3}{4}$
- the number 3 multiplied by $\frac{3}{4}$, i.e. scaling the number 3 to three-quarters of its size

and to see that the answer is the same. The first reinforces the idea that multiplication 'makes bigger' as the $\frac{3}{4}$ has, indeed, got bigger. However, in the second, the 3 has been made smaller when multiplied by the fraction.

Example 1: *Write the answer to each of these:*

a) $1 \times \frac{1}{3} =$ $\qquad$ $\frac{1}{3} \times 1 =$

b) $2 \times \frac{1}{3} =$ $\qquad$ $\frac{1}{3} \times 2 =$

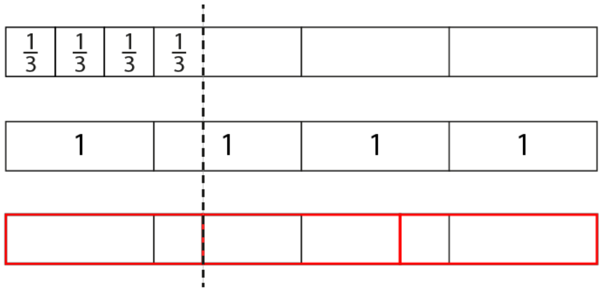
c) $4 \times \frac{1}{3} =$ $\qquad$ $\frac{1}{3} \times 4 =$

d) $7 \times \frac{1}{3} =$ $\qquad$ $\frac{1}{3} \times 7 =$

In *Example 1*, the fraction remains the same while the integer changes. Each part also has two versions of the same calculation. It is important to draw students' attention to the different ways we can think of multiplying. While working through this example, ask questions such as, *'What does each statement mean?'* and, *'Why do they give the same answer?'*

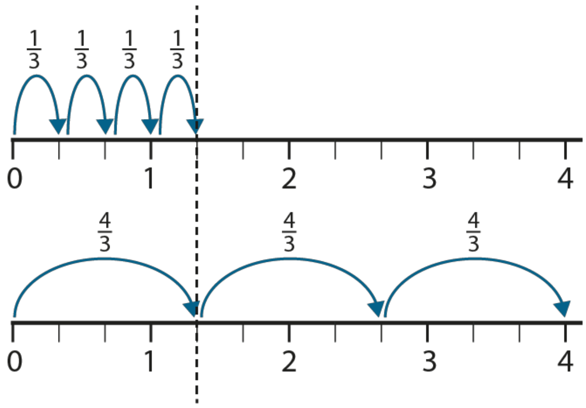
If the integer is the multiplier, then it is helpful to think of multiplication as 'groups of' (as in 'two groups of one third is two-thirds'). However, if the fraction is the multiplier, students are forced to re-think multiplication as scaling (as in 'one third of two'). This is important because when both numbers are fractions, multiplication cannot be thought of as 'groups of'. Students can be encouraged to draw diagrams to show such calculations.

For example, to show that $4 \times \frac{1}{3}$ and $\frac{1}{3} \times 4$ are the same:



The first bar model shows $\frac{1}{3}$ replicated four times (four groups of $\frac{1}{3}$). The third bar model shows how four (depicted in the second bar model) has been multiplied by one third (reduced to one third).

Students could also be encouraged to use number lines to show this equivalence:



Example 2: *Fill in the gaps with <, > or =.*

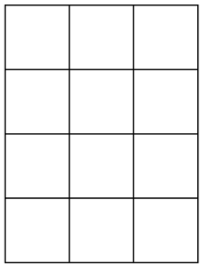
$\frac{1}{2} \times 4 \bigcirc \frac{1}{2}$ $\frac{1}{2} \times \frac{1}{2} \bigcirc \frac{1}{2}$

$\frac{1}{2} \times 2 \bigcirc \frac{1}{2}$ $\frac{1}{2} \times \frac{1}{4} \bigcirc \frac{1}{2}$

$\frac{1}{2} \times 1 \bigcirc \frac{1}{2}$

By comparing the relative sizes of one of the factors and the resulting product in *Example 2*, students' attention can be drawn to the point where the product becomes equal to and then less than $\frac{1}{2}$, and so develop their understanding of when the multiplier makes the product smaller. This example offers a context in which to explore the misconception that 'multiplication makes bigger'.

Example 3: *This is a picture for 4 × 3:*



Draw a similar picture for each of the following and use it to state each product.

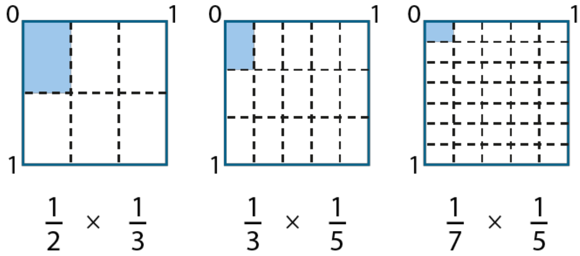
a)	4×2	*f)*	3×3	*k)*	1×1
b)	4×1	*g)*	2×3	*l)*	$\frac{1}{2} \times \frac{1}{3}$
c)	$4 \times \frac{1}{2}$	*h)*	1×3	*m)*	$\frac{1}{3} \times \frac{1}{5}$
d)	$4 \times \frac{1}{3}$	*i)*	$\frac{1}{2} \times 3$	*n)*	$\frac{1}{7} \times \frac{1}{5}$
e)	$4 \times \frac{1}{7}$	*j)*	$\frac{1}{5} \times 3$		

The purpose of *Example 3* is to understand how an area model can represent the multiplication of fractions. The numbers have been chosen to draw attention to the idea of reducing one side of the rectangle (while keeping the other side constant), and so reducing the area (product); and to focus on the fraction as the multiplier.

It will be important to pause after a few of these (during the first set of five and then again during the second set of five) and ask students, '*What is the same?*' and, '*What is different?*' about successive calculations, and, '*What do you notice about the product?*'

Part *k* could be done orally, with a student coming to the board and drawing a unit square. It will be important, at this stage, to introduce the idea of seeing each side as a unit that can be further reduced, or as a number line from 0 to 1 where fractions can be placed.

This important image and associated awareness will support students in seeing the multiplication of any two fractions as an area within the unit square, and hence in answering parts *l*, *m* and *n*:



Such diagrams may help students to make sense of the 'multiply the denominators' rule. It will be important to ask questions, such as, '*Why do you get sixths when you multiply halves and thirds?'* and *'Why do you get fifteenths when you multiply thirds and fifths?'* These might be generalised to consider what the denominator would be if fractions with denominator of *a* and denominator of *b* were to be multiplied.

Understand how to multiply unit, non-unit and improper fractions

Common difficulties and misconceptions: one common difficulty with multiplication of fractions is that the rule for doing this is very simple (multiply the numerators and multiply the denominators) and students often apply this rule without any understanding.

It will be important to use representations (for example, the area model) alongside the symbolic calculation in order to get a sense of what is happening when two fractions are multiplied together. In this way, students can make sense of the rule rather than just learning it without justification.

Students may also use inefficient methods if they are simply following the algorithm, for example, $\frac{4}{5} \times \frac{3}{4} = \frac{12}{20}$, and then need to cancel down without recognising that, in the original calculation, the 4 and the $\frac{1}{4}$ (the 4 being the numerator in the first fraction and denominator in the second) reduce to one and hence the answer must be $\frac{3}{5}$.

Example 1: *Complete these calculations.*

a) $\frac{1}{3} \times \frac{1}{5} =$	*b)* $2 \times \frac{1}{3} \times \frac{1}{5} =$	*c)* $\frac{2}{3} \times \frac{1}{5} =$
d) $\frac{1}{3} \times 2 \times \frac{1}{5} =$	*e)* $\frac{1}{3} \times \frac{2}{5} =$	f) $\frac{2}{3} \times \frac{2}{5} =$
g) $\frac{3}{4} \times \frac{3}{5} =$	*h)* $\frac{3}{4} \times \frac{4}{5} =$	*i)* $\frac{2}{3} \times \frac{3}{5} \times \frac{5}{7} =$

The intention of *Example 1* is for students to appreciate that non-unit fractions are integer multiples of unit fractions and that, by considering the commutative and associative laws, the multiplication of non-unit fractions can be derived.

To help enable this for part *a*, teachers could ask questions such as, *'Can you draw a picture to show this?'* and *'Can you see this as $\frac{1}{3}$ of $\frac{1}{5}$ and as $\frac{1}{5}$ of $\frac{1}{3}$?'* and ask similar 'commutativity' questions for parts *c* and *e*.

For parts *b* and *d*, it will be important to help students connect the answers to those for parts *c* and *e*. Some discussion of part *d* could include how the calculation might be read:

'Is it ($\frac{1}{3}$ × 2) × $\frac{1}{5}$?' 'Or is it $\frac{1}{3}$ × (2 × $\frac{1}{5}$) i.e. the same as $\frac{1}{3} \times \frac{2}{5}$?'

Students can then realise that these calculations can be seen as equivalent.

Parts *f* and *g* can be used as an opportunity to practise these ideas. The prompt *'In how many ways can you think of this calculation?'* can support students in seeing them in different ways to support fluency:
$\frac{2}{3}$ of $\frac{2}{5}$; $\frac{2}{5}$ of $\frac{2}{3}$; $2 \times \frac{1}{3} \times \frac{2}{5}$; $\frac{2}{3} \times 2 \times \frac{1}{5}$; $2 \times \frac{1}{3} \times 2 \times \frac{1}{5}$; $4 \times \frac{1}{3} \times \frac{1}{5}$, etc.
This sequence of different versions of the same calculation can be used to stimulate discussion and make sense of the 'multiply the numerators' rule. It will be important to encourage students to reason why this is true, using examples like the one above. Asking questions, such as the following, will support students' understanding.

'What is the same and what is different in these two calculations: $\frac{1}{3} \times \frac{1}{5}$ and $\frac{2}{3} \times \frac{2}{5}$?'

'Why is the answer to the second calculation four times bigger than the answer to the first calculation?'

Parts *h* and *i* provide opportunities for students to practise these ideas but with some added challenge (i.e. the possibility of cancelling before multiplying in part *h* and the extension to three fractions in part *i*).

The strategy of rewriting $\frac{2}{3} \times \frac{3}{5} \times \frac{5}{7}$ as $2 \times \frac{1}{3} \times 3 \times \frac{1}{5} \times 5 \times \frac{1}{7}$ and bracketing it as
2 × ($\frac{1}{3}$ × 3) × ($\frac{1}{5}$ × 5) × $\frac{1}{7}$ will help to reveal the structure and support students in understanding the process of 'cancelling' to simplify before calculating.

***Example 2**:*
Given that $\frac{8}{15} \times 465 = 248$ *find:*
$\frac{4}{15} \times 465$
$\frac{16}{15} \times 465$
$\frac{8}{3} \times 465$

In *Example 2*, students are invited to manipulate a given expression to find the answers to other calculations, further developing their understanding of the use of one fraction (such as a unit fraction) to find the product of two fractions.

On completion of the three given calculations, and having discussed their reasoning, students could then be asked to give some other results that they can find using the original product.

Example 3: *Which gives the greater result, calculation a or calculation b? Explain how you know.*

a) $\frac{12}{13} \times \frac{14}{15}$

b) $\frac{14}{15} \times \frac{15}{16}$

A key aspect of fluency with calculations of this type is to look for simplifications before multiplying numerators and denominators.

In *Example 3*, students should be encouraged to reason about the relative size of the answers by looking at the constituent parts of the product rather than calculating the product.