Key ideas

- Know the commutative law and use it to calculate efficiently
- Know the associative law and use it to calculate efficiently
- Know the distributive law and use it to calculate efficiently
- Calculate using priority of operations, including brackets, powers, exponents and reciprocals
- Use the associative, distributive and commutative laws to flexibly and efficiently solve problems*
- Know how to fluently use certain calculator functions and use a calculator appropriately

Exemplified significant key ideas

Understand the mathematical structures that underpin addition and subtraction of positive and negative integers

***Common difficulties and misconceptions**:* although students have been introduced to negative numbers at Key Stage 2, their experience is likely to have been set in a context, and it is unlikely that they will have carried out operations with negative numbers.

Students are now working with a new ‘type’ of number and, in doing so, challenging and extending their understanding of additive operations. Until the introduction of negative numbers, their experience will have been that addition makes larger and subtraction makes smaller. Including situations in in which this is not the case can be problematic.

When assessing students’ understanding of operations with negative numbers, Hart (1981) records that, when subtracting a positive integer from either a positive or negative number, many students simply subtract the numerals and then attempt to determine the sign for the answer. When subtracting a negative integer, many students used the rule that ‘two negatives make a positive’. Examining the structure of such calculations (using classroom examples, such as the ones offered below) rather than teaching such rules will help students overcome these difficulties.

***Example 1**: Fill in the blanks to make the calculations correct.*

a) *3 + 4 – 4 – ___ = 0*

b) *3 – 3 + 1 – ___ = 0*

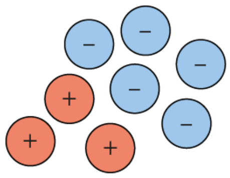
c) *15 + 7 – 15 = ___*

d) *10 – 7 – ___ = 0*

e) *182 – 82 – ___ = -1*

Example 1 is designed to draw students' attention to the way that pairs of numbers can be use so that an answer can be found without the need for calculating. In parts a, b and c, the pairing of numbers is clear. In part d, students need to identify that the calculation can be thought of as (3 + 7) − 7 − ___ = 0. It is awareness of this partitioning that is key.

Example 2: *These counters show −2.*



a) Another counter is added and the value changes to −1.
  (i) Was the counter a positive or a negative? Explain how you know.
  (ii) Write a number sentence to describe this.

b) One of the counters is taken away and the value changes to –1.
  (i) Was the counter a positive or a negative? Explain how you know.
  (ii) Write a number sentence to describe this.

c) Find two different ways to change the total of the counters to –3.
  (i) Explain how you know that you are correct.
  (ii) Write a number sentence to describe each of your strategies.

Students' understanding that numbers can be partitioned into zero pairs can be built on to see the equivalence of adding +1 and subtracting –1. A key point to be drawn out of *Example 2* is that the two operations give an equivalent result.

It is important that the symbols are used alongside the representation, and that the symbols are seen as describing the manipulations that have been made. This connection must be explicit if the representation is to support students' understanding of addition and subtraction with negative numbers.

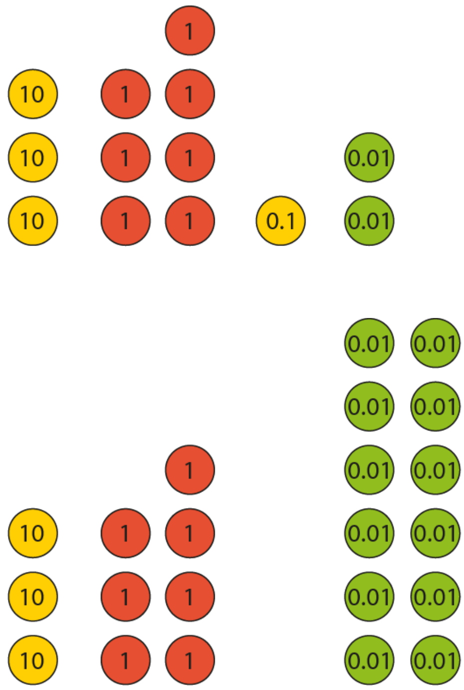
Generalise and fluently use addition and subtraction strategies, including columnar formats, with decimals

Common difficulties and misconceptions*:* students may be proficient with the standard column algorithms when adding and subtracting with integers, but using these methods with decimals can prove challenging to those students who do not understand the structures that underpin them. For example, when working with integers, students may be able to align the numbers being operated on 'from the right', but the inclusion of decimals means that this now needs to be refined to an understanding that the decimal points need to be aligned.

Aligning decimal points can also bring the additional challenge of using zero as a place holder. For example, when calculating 173.61 − 28.35082, students need to understand both the need for using zero as a place holder, and the equivalence of the given calculation and 173.61000 − 28.35082. Similarly, for examples such as 17 − 12.34 where there are 'no decimal places'.

The use of representations, such as place-value counters, can support students in understanding the structures that underpin the standard algorithms for both integers and decimal numbers. Examples are given below:

Example 1*: What is the same and what is different about these sets of place-value counters?*



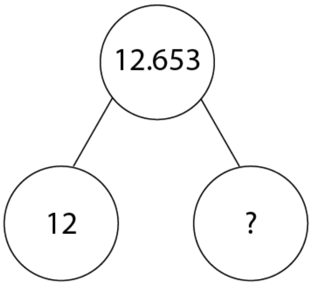
Place-value counters offer a flexible representation of multi-digit numbers; they allow students to see partitioning of the same number in different ways.

In *Example 1*, the number 37.12 is represented. By manipulating the counters, students can see that this is 3 × 10 + 7 × 1 + 1 × 0.1 + 2 × 0.01, but also that it is 3 × 10 + 7 × 1 + 12 × 0.01.

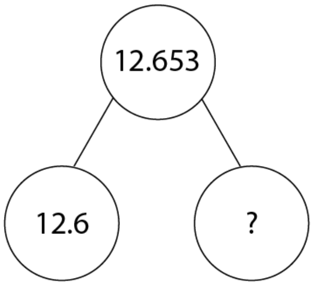
The understanding that the same number can be partitioned in these different ways is essential in efficient calculation, particularly when subtracting using the columnar method.

Example 2: *What are the missing numbers?*

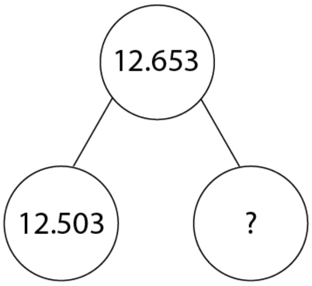
a)



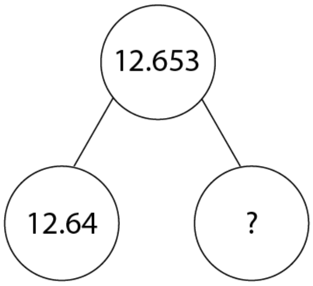
b)



c)



d)



Example 2 uses the part–part–whole model which students should be familiar with from Key Stages 1 and 2. This representation allows students to generalise their understanding of partitioning integers to partitioning with decimals.

Example 3: *Fill in the missing digits in these calculations.*

a)

	3	1	7	.	□	□	□
–	1	1	9	.	1	5	1
	□	□	□	.	8	1	1

b)

	□	□	□	.	2	5	3
–	1	4	9	.	1	5	1
	2	2	9	.	□	□	□

c)

	4	*0*	*3*	.	*1*	□	□
–	*3*	*1*	□	.	□	*2*	*5*
	□	□	*8*	.	*9*	*9*	*6*

In challenging students to fill in the blanks in *Example 3*, the routine algorithm is disrupted, and students need to think more deeply about the structures that underpin it; in particular, the process of exchange (for example, exchanging one ten for ten ones).

The questions here are sequenced so that the nature of the exchanges is varied, resulting in the complexity of the problem increasing.

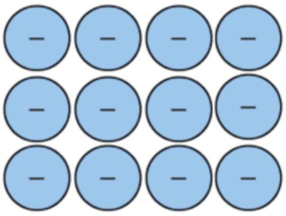
Understand the mathematical structures that underpin multiplication and division of positive and negative integers

Common difficulties and misconceptions: students are likely to have met negative numbers at Key Stage 2, but this will only have been in context, and they are unlikely to have carried out operations involving negative values.

Many Key Stage 3 students will have the misconception that ‘multiplication makes bigger’ when working with positive numbers. There may be additional layers to this when considering the difference between magnitude and value when working with negative numbers. For example, until students met negative numbers, their experience would be that numbers closer to zero are ‘smaller’ both in terms of value and magnitude. Hence students may think that −591 is larger than −3 because it’s further from zero, but agree that it’s a lower value. Modelling consistent language by using *‘greater than or less than’* rather than the more ambiguous *‘bigger or smaller’* will help with clarity.

Some students will come to working with negative numbers knowing informal rules such as ‘two negatives make a positive’. Exploring the limits of these generalities with a class and providing them with suitable representations will them to attain a deeper and more connected understanding. Examples are given below:

Example 1: *This diagram shows some counters in an array.*

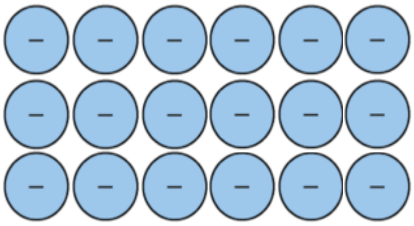


Robyn says, 'I can see three groups of negative four' and writes 3 × –4 = –12.

Taylor says, 'I can see four groups of negative three.'

a) Write a number sentence for Taylor's description of the array.

Sienna looks at another array



b) *Write two number sentences that describe this array.*

In *Example 1* an array is used to connect an existing representation for multiplication and extend it to include products with one negative factor.

Robyn and Taylor use the array to identify different groups of values. Referring to repeated addition and linking it to multiplication will help students to connect their understanding. For example, writing Robyn's statement as −4 + −4 + −4 = 3 ×−4 will extend this interpretation of multiplication to negative products.

Example 2: *Pete writes a calculation.*

−1 × (1 + −1) = −1 × 1 + −1 × −1

He says, 'This shows that negative one multiplied by negative one ***must*** *be positive one.'*

a) *What is the value of 1 + −1?*
b) *What is the value of −1 × (1 + −1)?*
c) *What is the value of −1 × 1?*
d) Do you agree with Pete that his calculation shows −1 × −1 = 1?

In *Example 2* students are asked to reason with two known properties (the zero property and the distributive law) to create a chain of reasoning that justifies why −1 × −1 = 1. This is a powerful argument that students should become familiar with to understand why this must be the case.

Generalise and fluently use written multiplication strategies to calculate accurately with decimals

Common difficulties and misconceptions: although students should be proficient with the standard written multiplication algorithm for calculating with whole numbers in Key Stage 2, students do not always make the connection with using the standard written multiplication algorithm for calculating with decimals in Key Stage 3. Some students will try to use the standard written algorithm with decimals. This can give the correct answer if the multiplier is a whole number but soon fails with decimal multipliers (see *Example 2*). Students can perceive multiplying decimals as 'new' learning rather than thinking about the underlying mathematical structure of a calculation such as:

$24.3 \times 1.2 = (243 \times 12) \times 0.1 \times 0.1$

or

$24.3 \times 1.2 = (243 \times 12) \div 10 \div 10$

Using *'If I know… then I also know…'* mathematical thinking empowers students to understand that if they know 243 × 12 = 2916 then they also can find the product of other calculations such as 24.3 × 1.2, 2.43 × 0.12, 0.243 × 1.2, etc. Avoiding mechanical practice of exclusively standard questions by varying the value of the multiplicand and multiplier is essential for students to form a rich understanding.

It is also useful to include some non-examples for students to critique and reason about as well. Students should be encouraged to estimate the product first to check they have transformed the calculation correctly using powers of 10. Examples are given below:

Example 1

Richard thinks

4.56 × 0.3 = 13.68 because

4.56

× 0.3

13.68

1 1

Nicola thinks:

2.5 × 1.1 = 27.5 because

2.5

× 1.1

2.5

25.0

27.5

Use estimation to decide whether Richard and Nicola are correct.

The calculations in *Example 1* support students' awareness that the standard written algorithm for multiplying whole numbers (short/long multiplication) can not be applied to calculations with decimals, in particular when the multiplier is a decimal number.

Students need to be encouraged to recognise that calculations such as 4.56 × 0.3 and 2.5 × 1.1 can be calculated as (456 × 3) ÷ 1000 and (25 × 11) ÷ 100 and to use extimation to check their answers.

Example 2: *Given 456 × 12 = 5 472 work out:*

a) 45.6 × 12

b) 4.56 × 12

c) 45.6 × 1.2

d) 4.56 × 1.2

e) 4.56 × 0.12

Example 2 draws attention to the digits in a calculation rather than the individual numbers. Asking '*what is the same and what is different?*' about the five calculations encourages students to notice the connections between the calculations and encourage an 'if I know… then I also know…' way of thinking.

Students' thinking can be deepened by asking more probing questions such as use 456 × 12 = 5 472 to find the products of other pairs of numbers.

Use the associative, distributive and commutative laws to flexibly and efficiently solve problems

Common difficulties and misconceptions*:* students' understanding of the laws of arithmetic is crucial if they are to be able to work flexibly to evaluate calculations.

A key idea here is that students are able to identify known facts, connections and relationships and use them to strategically simplify calculations. For some students, whose experience of mathematics may be that there is only one correct process that should be followed, this may prove challenging.

The strategy of inviting students to solve problems *'in as many different ways as you can'* can help to develop the skill of making sensible choices based on the numbers involved and the relationships between them. It is also helpful to choose examples that draw students' attention to certain useful connections and ask them, *'What do you notice?'*, for example, 9 999 + 999 + 99 + 9 + 5 or 2.75 × 5.4 + 27.5 × 0.46.

Example 1:

a) *Which of these is correct?*

$13 \times 99 = 10 \times 90 + 3 \times 9$

$13 \times 99 = 13 \times 100 - 13 \times 1$

$13 \times 99 = 13 \times 90 + 13 \times 9$

$13 \times 99 = 10 \times 99 + 3 \times 99$

$13 \times 99 = 15 \times 99 - 2 \times 99$

Which method do you prefer to calculate this product? Why?

b) *Which of these is correct?*

$19 \times 99 = 25 \times 99 - 6 \times 99$

$19 \times 99 = 20 \times 99 - 1 \times 99$

$19 \times 99 = 19 \times 100 - 19 \times 1$

$19 \times 99 = 10 \times 90 + 9 \times 9$

$19 \times 99 = 10 \times 90 + 9 \times 90$

$19 \times 99 = 19 \times 90 + 19 \times 9$

Which method do you prefer to calculate this product? Why?

c) *Which of these is correct?*

$77\,068 \div 5 \div 2 = (77\,068 \div 5) \div 2$

$77\,068 \div 5 \div 2 = 77\,068 \div (5 \div 2)$

$77\,068 \div 5 \div 2 = 77\,068 \div 2 \div 5$

$77\,068 \div 5 \div 2 = 77\,068 \div (5 \times 2)$

Which method do you prefer to calculate this quotient? Why?

d) *Which of these is correct?*

$7\,742 \div 14 = (7\,742 \div 7) \div 2$

$7\,742 \div 14 = (7\,742 \div 2) \div 7$

$7\,742 \div 14 = (7\,742 \div 10) \div 4$

Which method do you prefer to calculate this quotient? Why?

In *Example 1*, students are required to think flexibly to evaluate whether the different methods are correct or not, before choosing their preferred method for calculating the product or quotient.

The invalid methods each highlight and draw out a particular misconception, while the valid ones show a range of different possibilities.

It is important that students understand that while each of the valid methods gives a correct solution, some may be more efficient at reaching it than others. It is also worth noting that the most efficient method for one student may not be the most efficient for another. Students' preferences will depend on what facts they are able to recall fluently.

Example 2: *Are the following statements true or false? Explain how you know.*

a) $5 \times 3.2 + 3.2 \times 3 = 2.5 \times 4 \times 3.2$
b) $2 \times (17 \times 3.2 + 1.8 \times 17) = 17^2 - (7 \times 17)$

In *Example 2*, students are required to bring together the commutative, distributive and associative laws to simplify complex calculations.

A key awareness for students here is that some calculations can be simplified. Students should not automatically reach for their calculator. Instead, they should consider each calculation as a whole in order to identify relationships and possible known facts, so reducing the amount of calculation necessary. Rather than focus on the final result of each calculation, it will be more helpful to emphasise the laws of arithmetic that have been used to simplify the calculations.