Sequences

Overview

Students began to consider sequences in Key Stage 1, when step counting to learn times tables and when looking at the composition of numbers. In Key Stage 2, they were introduced to the use of symbols and letters to represent variables and unknowns in familiar mathematical situations and began to generalise number patterns.

Students will have explored non-numerical (shape) and numerical sequences, noticed a pattern, described the pattern in words and found the next term in the sequence from the previous term. They will primarily have focused on generating and describing linear number sequences, though they may have also experienced naturally occurring patterns in mathematics, such as square numbers.

The extent to which students have explored these concepts in depth may vary. Therefore, students should consolidate, secure and deepen their understanding of linear sequences and how to find and use term-to-term rules to generate the next term. Then, they can progress to describing any term in the sequence directly in relation to its position in the sequence.

This work extends students' knowledge of sequences through exploration of the mathematical structure, not just by spotting the patterns that the structure creates. Algebraic notation is used to express the structure, and students should become familiar with finding and using the nth term of a linear sequence. It is important that students have time to develop a full understanding of the connection between the notation and the sequence and come to see the nth term as a way of expressing the structure of every term in the sequence.

This learning has connections to other areas of algebra, particularly solving equations (when checking if a number is a term in a sequence) and straight-line graphs. Work on sequences both here and later in Key Stage 3 provides the foundation for exploring quadratic sequences and simple geometric progressions in Key Stage 4.

Prior learning

Before beginning sequences at Key Stage 3, students should already have a secure understanding of the following learning outcomes from study at upper Key Stage 2:

- Generate and describe linear number sequences
- Use simple formulae

and earlier in Key Stage 3:

- Understand multiples

- Understand integer exponents and roots
- Understand and use the conventions and vocabulary of algebra, including forming and interpreting algebraic expressions and equations

Checking prior learning

The following activities from the NCETM primary assessment materials offer useful ideas for teachers to use to check whether prior learning is secure.

Reference	Activity
Year 6 page 27	*Ramesh is exploring two sequence-generating rules.* *Rule A is: 'Start at 2, and then add on 5, and another 5, and another 5, and so on.'* *Rule B is: 'Write out the numbers that are in the five times table, and then subtract 2 from each number.'* *What's the same and what's different about the sequences generated by these two rules?*
Year 6 page 27	*On New Year's Eve, Polly has £3.50 in her money box. On 1 January she puts 30p into her money box. On 2 January she puts another 30p into her money box. She continues putting in 30p every day.* *How much money is in the box on 10 January?* *How much money is in the box on 10 February?* *Write a sequence-generating rule for working out the amount of money in the money box on any day in January.*

Language

arithmetic sequence, *n*th term of a sequence, sequence, term

Progression through key ideas

Understand the features of a sequence

Students should be familiar with finding and using difference patterns in linear sequences to generate the terms of a sequence. This collection of key ideas provides an opportunity to secure and deepen understanding by applying this prior learning to other linear and non-linear sequences, including those containing negative and fractional terms.

Key ideas

- Appreciate that a sequence is a succession of terms formed according to a rule*
- Understand that a sequence can be generated and described using term-to-term approaches
- Understand that a sequence can be generated and described by a position-to-term rule

Recognise and describe arithmetic sequences

Students should realise that generating many terms in a sequence using the term-to-term rule is not an efficient method. By exploring the mathematical structure of a linear sequence, students can instead describe linear sequences using the nth term. Students should then experience a range of ascending and descending sequences, and those containing terms that are decimals and fractions, in order to develop a deep and secure understanding of the *n*th term and how to use it.

The nth term is new learning to Key Stage 3. It is crucial that students are given time to become fluent at describing linear sequences using the nth term rule, as well as reason with and apply it in order to solve mathematical problems, such as finding the 50th and 100th terms of a linear sequence.

Key ideas

- Understand the features of an arithmetic sequence and be able to recognise one
- Understand that any term in an arithmetic sequence can be expressed in terms of its position in the sequence (*n*th term)*
- Understand that the *n*th term allows for the calculation of any term
- Determine whether a number is a term of a given arithmetic sequence

Exemplified significant key ideas

Appreciate that a sequence is a succession of terms formed according to a rule

Common difficulties and misconceptions: students may have an intuitive sense that the terms in a sequence progress logically according to a rule but may find it difficult to express clearly what that rule is.

They should, through discussion and sharing ideas, be able to clearly articulate a rule and should be encouraged to use mathematical language to describe sequences wherever possible. For example, when describing the sequence 3, 5, 7, 9, 11, … students may often say, ‘It goes up in 2s’. Through discussion, this response can be refined so that students are more explicit regarding the starting number and the amount added each time. For example, ‘The sequence begins with three, and two is added each time' or 'The first term is three, the second term is three plus two, the third term is three, plus two, plus two, etc.'.

It is not uncommon for students to notice only additively increasing sequences (i.e. arithmetic sequences where the common difference is positive), so they should experience a varied collection of types of sequence. For example:

- 23, 19, 15, 11, 7, 3, … (decreasing arithmetic sequence)
- 3, 6, 12, 24, 48, 96, … (geometric sequence, where there is a constant multiple or ratio between successive terms)
- 1, 4, 5, 9, 14, 23, … (Fibonacci-like sequence, where terms are generated by adding the two preceding terms)

Also, sequences of squares (or cubes or multiples of a given number or odd numbers) are useful in that, while students may wish to describe them using differences, there is the opportunity to notice that, for example, 'the third number in the sequence is the third square (or cube or multiple of seven or odd number)'.

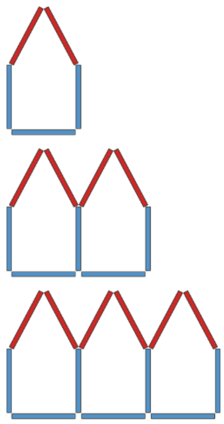
The focus in this key idea is being aware that there is a consistent mathematical rule that generates terms, without necessarily describing that rule precisely. However, discussion of different sequences and how they progress, using increasingly sophisticated mathematical language, is an important precursor to finding term-to-term and position-to-term rules later.

Students should also experience sequences where there are multiple ways in which the sequence could be extended. For example, the terms in the sequence 1, 2, 4, … could be generated by:

- doubling each term to get the next (1, 2, 4, 8, 16, 32, …)
- adding one, then two, then three, etc. (1, 2, 4, 7, 11, …).

Examples are given below.

Example 1: *Here is a sequence of shapes.*



What is staying the same?

What is changing?

a) *How many red sticks will the 4th shape of this sequence require?*
b) *How many blue sticks will the 5th shape of this sequence require?*
c) *What can you say about the 18th shape in this sequence?*
d) *How many blue sticks are in the shape that has 16 red sticks?*

Example 1 offers a sequence of 'growing shapes', which can be helpful in supporting students to spot patterns and rules.

Students may be more readily able to see (particularly if pictures are colour-coded, as they are here) both how the number of sticks is increasing and how each picture/term has the same structure, i.e. always having a number of pairs of red sticks (depending on the position number) and one more than that number of blue sticks.

Example 2: *Afsal thinks that the following four statements about sequences are true:*

a) *Sequences always increase.*
b) *A sequence can have either positive or negative terms, but it cannot have both.*
c) *An arithmetic sequence always has a common difference that is an integer.*
d) *All sequences have terms that increase by the same amount each time.*

For each statement, find an example of a sequence which shows that Afsal is not correct.

In *Example 2*, students are asked to generate examples of their own that fit certain criteria, which can be an effective way of encouraging deeper thinking.

An example also features an important use of variation: encouraging students to be aware of the full range of examples of a certain mathematical object (in this case, sequences) – both standard and non-standard examples.

Students often develop misconceptions as a result of being exposed to a limited range of examples. This example is designed to get students to think about such possible misconceptions and refine in their own minds what makes a list of numbers form a sequence.

Example 3: *Here is a sequence of numbers.*

5, 10, 15, 20, 25, 30, …

a) *Will the number 67 be in the sequence?*

   Explain your answer.

b) *What position would 55 be in the sequence?*

   Give a reason for your answer.

Give some more examples of numbers that are (and are not) in this sequence and explain why.

Example 3 encourages students to think about what structure is common to each term in the sequence. Students will often look to see how the sequence is progressing from term to term, rather than looking for this commonality.

Being able to say that ‘All of the numbers are multiples of five’ (and not just that ‘They go up in fives’) is an important step as it prepares the ground for the important question ‘Which multiple of five is each term?’. This will support students later in being able to identify any term in the sequence.

Here are some other sequences that students could work on in a similar way:

- 5, 15, 25, 35, …
- 90, 80, 70, 60, …
- $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
- 0.3, 0.6, 0.9, 1.2, …

Understand that any term in an arithmetic sequence can be expressed in terms of its position in the sequence (*n*th term)

Common difficulties and misconceptions: when examining arithmetic sequences where each term is obtained by adding a fixed amount (constant difference) to the previous term, it is natural for students to express the rule in terms of this fixed amount. For example, students may see the sequence 4, 7, 10, 13, … as ‘add three’ and think that, therefore, the nth term is $n + 3$ rather than the correct $3n + 1$.

This misconception can be explored by examining the three times table (i.e. 3 (3 × 1), 6 (3 × 2), 9 (3 × 3), 12 (3 × 4), 15 (3 × 5), …, 3n) and a range of other sequences that consist of terms in the three times table with one, two, three, etc. added (or subtracted). Students should be encouraged to notice what is the same and what is different – i.e. it is the ‘$3n$’ that determines the ‘increasing by three’. It will then be helpful for students to experience substituting n = 1, 2, 3, etc. into the expression ‘$n + 3$’ to realise that this will give a sequence which begins at four and increases by one, rather than beginning at four and increasing by three.

The fundamental awareness here is that, in the general statement $3n + 1$, the common difference is represented by the ‘3’ (the coefficient of n) because as n increases, the value of the whole expression increases by three. It will be important to explore the potential confusion between multiples of three, for example, and numbers which increase by three, and to recognise that these are not necessarily the same thing. Examples are given below.

***Example 1**: The terms in each of these sequences are generated by adding 3 each time and the 17th term is circled.*

a) *4, 7, 10, 13, 16, …, 52*

b) *5, 8, 11, 14, 17, …, 53*

c) *14, 17, 20, 23, 26, …, 62*

d) *27, 30, 33, 36, 39, …, 75*

For each sequence, write the 17th term as an expression involving 17.

By keeping the constant difference the same in all parts of *Example 1*, it is intended that students will see that three is the multiplier in the 17th term, because three is added each time. By the 17th term, seventeen 3s will have been added.

It may help to draw students' attention to the fact that, as each term has been increased by three, it is possible to write each term in such a way to show the '3's:

4 = 1 + 3

7 = 1 + 3 + 3

10 = 1 + 3 + 3 + 3, etc.

and to ask, '*How many 3s will have been added to generate the 17th term?*'

It would be beneficial to probe students' understanding a bit deeper by asking for the 25th (or 50th, or 100th, or 347th, etc.) term and then to generalise this to the *n*th term.

***Example 2:** How do these sequences increase or decrease? How can you tell just by looking at the nth term?*

a) $5n + 3$
b) $-7n + 3$
c) $0.9n - 3$
d) $-n + 3$

Example 2 offers an opportunity to look at how the coefficient of *n* affects the sequence. Students often encounter examples that predominantly feature increasing sequences, so both increasing and decreasing sequences are included here. To support their developing understanding, it will be important for students to articulate clearly what happens to each term as *n* increases and to realise that the coefficient of *n* determines the increase value.

Sentences of the form '*As the value of n increases by one, the value of each term increases (or decreases) by _*' should be encouraged.

Example 3:

a) *Mo thinks the nth term of the sequence*

10, 6, 2, −2, … is 4n + 6.

Do you agree? Explain your reasoning.

b) *Olivia thinks the nth term of the sequence*

2, 7, 12, 17, … is n + 5.

Explain why Olivia is incorrect.

c) *0, 5, 10, 15, 20, …*

10, 15, 20, 25, ….

−5, 0, 5, 10, 15, …

−25, −20, −15, 10, −5, …

Liam thinks all the sequences can be described using the nth term ‘5n’ as they all ‘increase by 5’.

Do you agree? Explain your answer.

d) *True or false?*

The 10th term of the sequence 3, 7, 11, 15, 19, … is 38.

In *Example* 3, variation is used to address some common misconceptions. An aspect of variation is to vary ‘what it’s not’ (as well as what it is) to help students clarify the idea in their minds. Part a is designed to encourage students to notice that although the difference between the terms is four, the term-to-term rule is ‘subtract four’ rather than ‘add four’. The rule $4n + 6$ will generate the 1st term correctly but will not generate subsequent terms accurately.

Parts b and c encourage students to pay attention to the position of each term in the sequence and not the term-to-term rule.

Part d highlights the misconception that the 10th term is twice the 5th term.